ISRAEL JOURNAL OF MATHEMATICS 89 (1995), 77–134

THE POISSON BOUNDARY OF COVERING MARKOV OPERATORS

BY

VADIM KAIMANOVICH*

IRMAR, Université de Rennes-1, Rennes 35042, France
e-mail: kaimanov@univ-rennes1.fr

ABSTRACT

Covering Markov operators are a measure theoretical generalization of both random walks on groups and the Brownian motion on covering manifolds. In this general setup we obtain several results on ergodic properties of their Poisson boundaries, in particular, that the Poisson boundary is always infinite if the deck group is non-amenable, and that the deck group action on the Poisson boundary is amenable. For corecurrent operators we show that the Radon–Nikodym cocycles of two quotients of the Poisson boundary are cohomologous iff these quotients coincide. It implies that the Poisson boundary is either purely non-atomic or trivial, and that the action of any normal subgroup of the deck group on the Poisson boundary is conservative. We show that the Poisson boundary is trivial for any corecurrent covering operator with a nilpotent (or, more generally, hypercentral) deck group. Other applications and examples are discussed.

0. Introduction

The classic **Poisson formula** giving an integral representation of a bounded harmonic function in the unit disk in terms of its boundary values has a long history (as it follows from its very name). For an arbitrary Markov operator $P: L^\infty(X,m)\hookleftarrow$ on a measure space (X,m) one can define its **Poisson boundary** as a measurable space Γ with a **harmonic measure type $\boldsymbol{\nu}$** on it such that the space of **bounded harmonic functions** $H^\infty(X,m,P)$ =

* Supported by a British SERC Advanced Fellowship. A part of this work was done during my stay at MSRI, Berkeley supported by NSF Grant DMS 8505550.
Received April 26, 1993

$\{f \in L^\infty(X,m)\colon Pf = f\}$ is isometric to the space $L^\infty(\Gamma,\boldsymbol{\nu})$. This correspondence between functions $f \in H^\infty(X,m,P)$ and their boundary values $\widehat{f} \in L^\infty(\Gamma,\boldsymbol{\nu})$ has the following property: for any initial probability distribution $\theta \prec m$ there exists a probability measure $\nu_\theta \prec \boldsymbol{\nu}$ such that $\langle\theta, f\rangle = \langle\nu_\theta, \widehat{f}\rangle$ (the generalized **Poisson formula**).

This isometry provides a convenient language for describing the space of bounded harmonic functions. In particular, absence of bounded harmonic functions (the **Liouville property**) is equivalent to triviality of the Poisson boundary, the space of bounded harmonic functions is finite-dimensional iff the Poisson boundary is finite, and atoms in the Poisson boundary are in one-to-one correspondence with bounded minimal harmonic functions.

The Poisson boundary describes **stochastically significant** behaviour of the corresponding Markov chain at infinity. Unlike the **Martin boundary**, the Poisson boundary is defined in measure theoretical terms only (as the space of ergodic components of the shift in the unilateral path space of the corresponding Markov chain), so that it does not require any topology on the state space. In the case when the Martin boundary of the operator P is well defined (or, at least, when transition probabilities of the operator P are absolutely continuous and the space of **positive harmonic functions** is a simplex [Dy]), the Poisson boundary coincides with the Martin boundary considered as a measure space with the representing measure of the constant harmonic function.

Markov operators corresponding to **random walks on groups** were the first large class of Markov operators for which the Poisson boundary was profoundly studied (see [Fu1], [Fu2], [Fu3], [Gu1], [Gu2], [KV], [K3], [K7], [Ra]). In this situation the Markov operator is invariant with respect to the left action of the group on itself, so that the Poisson boundary is also endowed with a natural action of the group. The next step was to consider the Poisson boundary of the Markov operator corresponding to the Brownian motion on a covering Riemannain manifold. Here the operator is invariant with respect to the action of the deck transformations group of the cover, so that once again the Poisson boundary is endowed with a group action [LS], [K4].

Since the Poisson boundary is a measure theoretical construction, it is natural to look at the Poisson boundary of covering Riemannian manifolds from a measure theoretical point of view. It leads to the notion of a **covering Markov operator** $\widetilde{P}$. Its state space $(\widetilde{X},\widetilde{m})$ is a **covering measure space**, i.e., it is endowed with a measure preserving completely dissipative action of a countable **deck group** G, and the operator $\widetilde{P}$: $L^\infty(\widetilde{X},\widetilde{m})\hookleftarrow$ is invariant with respect to the action of G. Thus, it defines the **quotient Markov operator** on the quotient measure space $(X,m)=(\widetilde{X},\widetilde{m})/G$. This definition embraces both random walks on countable groups and diffusion processes on covering manifolds (or, random walks on covering graphs). In the former case the quotient space is a singleton, and the quotient operator is trivial, whereas in the latter case the quotient Markov operator corresponds to the Brownian motion on the quotient manifold. Invariant Markov operators on Lie groups and their homogeneous spaces can also be considered as covering Markov operators with respect to the action of a corresponding discrete subgroup.

The aim of this paper is to carry out consistently the measure theoretical approach for studying the Poisson boundary of covering Markov operators. This point of view turns out to be quite productive. In particular, we generalize and strengthen all results on the Poisson boundaries of covering Riemannian manifolds from [LS].

The structure of the paper is the following.

Section 1 is devoted to basic definitions and facts connected with general Markov operators and their Poisson boundaries.

In Section 2 we introduce covering Markov operators and state several basic properties of their Poisson boundaries connected with **amenability**. The Poisson boundary of any covering operator with a non-amenable deck group is infinite (Theorem 2.2.3), and the stabilizer subgroup of a.e. point from the Poisson boundary is amenable (Theorem 2.2.4). Proofs of these results make use of conditional operators corresponding to the points of the Poisson boundary. Since the Poisson boundary can be defined as the **Mackey range** of a cocycle on the path space of the corresponding Markov chain, the action of the deck group on the Poisson boundary of a Markov operator is amenable (Theorem 2.3.3).

In §2.4 we consider **corecurrent** Markov operators (i.e., such covering operators that their quotient operator is recurrent). For any Markov operator P with a stationary measure m one can define its **Poisson extension** P^∞ by mutiplying the state space of the operator P by the Poisson boundary $\Gamma(P)$, so that the paths of the Markov chain corresponding to the Poisson extension have the form $(x_n, \mathbf{bnd}\overline{x})$, where $\mathbf{bnd}\overline{x}\in\Gamma$ is the point of the Poisson boundary corresponding to the path $\overline{x}=(x_n)$. The operator P^∞ has invariant measure $dm(x)d\nu_x(\gamma)$, where ν_x is the harmonic measure on the Poisson boundary corresponding to a point x. The Poisson boundary of the Poissson extension P^∞ coincides with the Poisson boundary of P, and the Poisson boundary of the reversal of P^∞ coincides with the product of Poisson boundaries of the operator P and its reversed operator (Theorem 1.4.3). The main technical tool used for studying corecurrent operators is Theorem 2.4.5: if a covering Markov operator is corecurrent, then its Poisson extension is also corecurrent. It immediately implies ergodicity of the deck group action on the product of the Poisson boundaries of a corecurrent operator and its reversed operator (Theorem 2.4.6).

In Section 3 we consider corecurrent Markov operators with an absolute continuity condition (P): for almost all points x from the state space their harmonic measures ν_x are absolutely continuous with respect to the harmonic measure type $\boldsymbol{\nu}$ on the Poisson boundary (there exists a **Poisson kernel**). This condition is, in particular, satisfied for all Harris corecurrent operators. If, in addition, almost all measures ν_x are **equivalent** to the harmonic measure type $\boldsymbol{\nu}$ (this condition can be considered as a very weak form of the **Harnack inequality at infinity**), then we show that the **Radon–Nikodym cocycles** on the Poisson boundary have the following **rigidity type property**: the cohomology class of the Radon–Nikodym cocycle of a G-invariant quotient of the Poisson boundary determines this quotient (Theorem 3.2.1). As a corollary we obtain that conditional measures corresponding to any two distinct G-invariant partitions $\xi \prec \zeta$ of the Poisson boundary are purely non-atomic (Theorem 3.3.1). Thus, the Poisson boundary is either trivial or purely non-atomic, and the action of any normal subgroup on the Poisson boundary is conservative (Theorem 3.3.3). So, any finite normal subgroup of the deck group acts trivially on the Poisson boundary. These results are new even for the Poisson boundary of random walks on discrete groups. In particular, they explain why non-trivial finite covers connected with the Poisson boundary of semi-simple Lie groups [Fu1] can arise only in the situation when the corresponding random walk is not irreducible.

Theorem 3.3.1 implies that the Poisson boundary of a corecurrent covering Riemannian manifolds is either trivial or purely non-atomic ($\equiv$ all minimal harmonic functions are unbounded). This property has been already known for covers of finite volume manifolds [K4] and leaves of Riemannian foliations with a transversally invariant measure [K5] where it was obtained by entropy methods.

Section 4 is devoted to application of general methods to more concrete situations. In §4.1 we show that the **center** of the deck group acts trivially on the Poisson boundary of a corecurrent Markov operator $\widetilde{P}$ (Theorem 4.1.1). Our proof does not use any Harnack type argument, and is based on the uniqueness of the stationary measure $d\widetilde{m}(x)d\nu_x(\gamma)$ of the Poisson extension of the operator $\widetilde{P}$. If $c \in Z(G)$, then the measure $d\widetilde{m}(x)d\nu_{cx}(\gamma)$ is also stationary, which implies that a.e. $\nu_x = \nu_{cx}$, so that the action of c is trivial. By transfinite induction it implies triviality of the Poisson boundary for corecurrent operators with **hypercentral** (in particular, **nilpotent**) deck groups (Theorem 4.1.4). Our argument also implies the (well-known) triviality of the Poisson boundary of random walks determined by spread-out probability measures on Lie groups with nilpotent lattices. Note that the question about the triviality of the Poisson boundary of random walks on such groups determined by an arbitrary singular measure (or, more generally, of general corecurrent covering operators with a nilpotent deck group) is still open (see [Gu1], [Gu2]).

Another application (§4.2) is to **conformal densities of divergence type groups** of hyperbolic motions. If the critical exponent δ satisfies the inequality $\delta \geq d/2$, then the conformal density is the harmonic measure of a corecurrent diffusion process on $\mathbb{H}^{d+1}$, so that results of Section 3 imply the rigidity of the corresponding Radon–Nikodym cocycles (Theorem 4.2.4). In particular, the action of any normal subgroup is conservative with respect to the conformal density. For the case $\delta = d$ (i.e., when the quotient manifold $\mathbb{H}^{d+1}/G$ is recurrent) this result was earlier obtained by Velling and Matsuzaki [VM].

In §4.3 we give simple examples of cotransient covering Markov operators with purely atomic Poisson boundary.

In Section 5 we consider interrelations between general covering operators and the simplest possible covering operators which correspond to **random walks on countable groups**. In §5.1 we show that for two classes of corecurrent operators (operators on a discrete state space and operators corresponding to **diffusion processes**) their Poisson boundary coincides with the Poisson boundary of an appropriate random walk on the deck group, so that the covering operator is in a sense approximated by the random walk on the deck group. In the proof we use the Furstenberg–Lyons–Sullivan construction of the approximating random walk [Fu2], [LS].

Considering random walks on groups instead of general corecurrent operators makes proofs of results from Sections 2, 3 and 4 much easier. For reader's convenience we give these proofs in §5.2. Also, looking first at these proofs would simplify understanding main ideas of Sections 2, 3 and 4. In view of §5.1, this generality is sufficient to deal with the Poisson boundaries of corecurrent diffusion processes, so that the reader interested in applications to Riemannian manifolds can skip the bulk of the paper and read Section 5 only.

1. The Poisson boundary

In this Section we shall introduce the basic notions and definitions connected with the (measurable) boundary theory of Markov operators. The main references concerning the general theory of Markov operators are [Fo], [Re] and [Kr]. Following [K7], we define the Poisson boundary of a Markov operator as the space of ergodic components of the shift in its unilateral path space (see also [Dy], [De] for a discussion of the boundary theory). The only non-standard notion introduced here is that of the Poisson extension P^∞ of a Markov operator P (§1.4). Its state space is the product of the state space of the original operator and its Poisson boundary. The Poisson boundary of the Poisson extension coincides with the Poisson boundary of the operator P, and the Poisson boundary of the reversed operator $\check{P}^\infty$ is the product of the Poisson boundaries of the operator P and its reversed operator $\check{P}$ (Theorem 1.4.3).

1.1 Markov operators

1.1.1 Definition: A linear operator P: $L^\infty(X,m)\hookleftarrow$ in a σ-finite measure space
(X,m) is called **Markov** if

(1) P preserves positivity, i.e., $Pf \geq \mathbf{0}$ for any function $f \geq \mathbf{0}$;
(2) P preserves constants, i.e., $P\mathbf{1} = \mathbf{1}$ for the function $\mathbf{1}(x) \equiv 1$;
(3) P is continuous in the sense that $Pf_n \downarrow \mathbf{0}$ a.e. whenever $f_n \downarrow \mathbf{0}$ a.e.

1.1.2 The adjoint operator P^* of a Markov operator P: $L^\infty(X,m)\hookleftarrow$ acts in the space of integrable functions on the space (X,m), or, in other words, in the space of measures on X absolutely continuous with respect to m. We shall use the notation θP for the measure on X with the density $P^*(d\theta/dm)$, so that

$$\langle \theta P, f\rangle = \langle \theta, Pf\rangle \qquad \forall f \in L^\infty(X,m) .$$

1.1.3 A (σ-finite) initial distribution $\theta \prec m$ gives rise to a Markov measure ${}_\theta\mathbf{P}^{\mathbb{Z}_+}$ in the (unilateral) **path space** $X^{\mathbb{Z}_+}$ of the **associated Markov chain** on X with **one-dimensional distributions** θP^n. The **unilateral shift** T^+ in the path space $X^{\mathbb{Z}_+}$ acts on the measure ${}_\theta\mathbf{P}^{\mathbb{Z}_+}$ as

$$T^+({}_\theta\mathbf{P}^{\mathbb{Z}_+}) = {}_{\theta P}\mathbf{P}^{\mathbb{Z}_+} .$$

1.1.4 Let π_x, $x \in X$ be the **one-step transition probabilities** of the operator P. The probability measures π_x are not necessarily absolutely continuous with respect to m, and can be defined for almost all points $x \in X$ as conditional measures of the measure ${}_\theta\mathbf{P}^{\mathbb{Z}_+}$, where $\theta \sim m$ is a certain reference probability measure on X. [All measure spaces in this paper are assumed to be **Lebesgue spaces**, so that the conditional decomposition always exists and is unique (mod 0) — see below §3.1.] In terms of the measures π_x the operator P has the form

$$Pf(x) = \int f(y)\, d\pi_x(y) .$$

1.1.5 Definition: A measure m is a **stationary** measure of a Markov operator P if

$$(\mathrm{Stat}_1) \qquad mP = m ,$$

or, equivalently,

$$(\mathrm{Stat}_2) \qquad \langle m, Pf\rangle = \langle m, f\rangle \qquad \forall f \in L^\infty(X,m) .$$

1.1.6 The adjoint operator P^* can be extended to a Markov operator $\check{P}$ in the space $L^\infty(X,m)$ if and only if $mP = m$. In this situation the operator $\check{P}$ is called the **backward** (or, **reversed**) operator corresponding to the **forward** operator P (with respect to the stationary measure m). If $P = \check{P}$, then the operator P is called **reversible**.

Stationarity of the measure m means that the measure ${}_m\mathbf{P}^{\mathbb{Z}_+}$ in the path space $X^{\mathbb{Z}_+}$ is invariant with respect to the unilateral shift T^+, so that it can be naturally extended to a measure ${}_m\mathbf{P}^{\mathbb{Z}}$ in the space of **bilateral paths** $X^{\mathbb{Z}}$ invariant with respect to the **bilateral shift** T.

The measure m is also a stationary measure of the reversed operator, and the bilateral path space of the operator $\check{P}$ (with the same one-dimensional stationary distribution m) can be obtained from the bilateral path space $(X^{\mathbb{Z}}, {}_m\mathbf{P}^{\mathbb{Z}})$ of the operator P by the **time reversion** $\{x_n\} \mapsto \{x_{-n}\}$. It reflects the fact that the **Markov property** — independence of the future and the past provided the present is fixed — does not depend on the chosen direction of time, so that the time reversal of a Markov measure is also Markov.

1.1.7 If for almost all points $x \in X$ the transition probabilities π_x have densities with respect to the measure m

$$p(x,y) = \frac{d\pi_x}{dm}(y) ,$$

i.e.,

$$Pf(x) = \int f(y)p(x,y)\, dm(y) ,$$

then the measure m is stationary if and only if $\int p(x,y)\, dm(x) \equiv 1$, and in this case the transition densities $\check{p}(\cdot,\cdot)$ of the reversed operator $\check{P}$ with respect to the measure m are given by the formula $\check{p}(x,y) = p(y,x)$. In particular, reversibility of P means that the transition densities $p(\cdot,\cdot)$ are symmetric.

1.1.8 Below we shall need the following elementary fact.

PROPOSITION: *Let P: $L^\infty(X,m)\hookleftarrow$ be a Markov operator satisfying condition* (Stat). *Then a measure $m' \prec m$ is a stationary measure of the reversed operator $\check{P}$ if and only if its density $\varphi = dm'/dm$ is a P-harmonic function, i.e., $P\varphi = P$.*

Proof: Indeed, stationarity of the measure m' means that

$$\langle m', \check{P}f\rangle = \langle m', f\rangle \qquad \forall f \in L^\infty(X,m) ,$$

i.e.,

$$\langle \varphi, \check{P}f\rangle_m = \langle \varphi, f\rangle_m \qquad \forall f \in L^\infty(X,m) ,$$

so that

$$\langle P\varphi, f\rangle_m = \langle \varphi, f\rangle_m \qquad \forall f \in L^\infty(X,m) ,$$

the latter property being equivalent to the equality $P\varphi = \varphi$. ∎

1.2 The Poisson boundary

1.2.1 Definition: Let $P: L^\infty(X,m)\hookleftarrow$ be a Markov operator. The space $\Gamma = \Gamma(X,m,P)$ of the **ergodic components** of the time shift T^+ in the unilateral path space $(X^{\mathbb{Z}_+}, {}_m\mathbf{P}^{\mathbb{Z}_+})$ is called the **Poisson boundary** of the operator P. Denote by **bnd**: $X^{\mathbb{Z}_+} \to \Gamma$ the corresponding quotient map, and by $\nu_\theta = \mathbf{bnd}({}_\theta\mathbf{P}^{\mathbb{Z}_+})$ the **harmonic measure** on $\Gamma(X,m,P)$ corresponding to an initial probability distribution $\theta \prec m$. The equivalence class $\boldsymbol{\nu}$ of harmonic measures ν_θ, $\theta \sim m$ is called the **harmonic measure class** on Γ.

1.2.2 By the definition, the space $L^\infty(\Gamma, \boldsymbol{\nu})$ of bounded measurable functions on the Poisson boundary is isometric to the subspace of $L^\infty(X^{\mathbb{Z}_+}, {}_m\mathbf{P}^{\mathbb{Z}_+})$ consisting of shift-invariant functions. The harmonic measures ν_θ satisfy the identity

$$\nu_\theta = \nu_{\theta P}, \qquad \nu \prec m ,$$

and there is a canonical isometry $f \leftrightarrow \widehat{f}$ between the space

$$H^\infty(X,m,P) = \{f \in L^\infty(X,m): Pf = f\}$$

of bounded **harmonic functions** of the operator P and the space $L^\infty(\Gamma, \boldsymbol{\nu})$ such that

$$\langle \theta, f \rangle = \langle \nu_\theta, \widehat{f} \rangle \qquad \forall\, \theta \prec m .$$

Thus, triviality of the Poisson boundary ($\equiv$ ergodicity of the shift in the unilateral path space) is equivalent to absence of non-constant bounded P-harmonic functions (the **Liouville property**).

1.2.3 Let $\{\nu_x\}$ be the family of **harmonic measures** on the Poisson boundary Γ corresponding to points $x \in X$. These measures are defined as conditional measures of a fixed reference probability measure ν_θ, $\theta \sim m$, and are not necessarily absolutely continuous with respect to the harmonic measure type $\boldsymbol{\nu}$. For any probability measure $\theta \prec m$

$$\nu_\theta = \int \nu_x \, d\theta(x) ,$$

and in terms of the measures ν_x the **Poisson formula** takes the form

$$f(x) = \langle \nu_x, \widehat{f} \rangle .$$

1.2.4 A subset $A \subset X$ is called **invariant** for the operator P if the function $\mathbf{1}_A$ is P-harmonic, i.e., $P\mathbf{1}_A = \mathbf{1}_A$. Invariance of a set A means that A and its complement $X \setminus A$ do not communicate, i.e., the function $\mathbf{1}_A$ is constant along a.e. path. An arbitrary Markov operator can be (uniquely) decomposed into irreducible components by factorizing its state space by the measurable partition corresponding to the σ-algebra of invariant sets. Thus, below we can without any loss of generality assume that the operator P satisfies the following **irreducibility** condition

(Irr) The operator P has no non-trivial invariant sets.

1.2.5 The following useful result is essentially the "zero" part of one of the Derriennic's 0-2 laws [De], [K7].

PROPOSITION: *Let P: $L^\infty(X, m) \hookleftarrow$ be a Markov operator, and $\theta, \theta' \prec m$ be two probability measures on X. Then*

$$\|\nu_\theta - \nu_{\theta'}\| = \lim_{n\to\infty} \frac{1}{n+1} \left\| \sum_{k=0}^{n} (\theta - \theta')P^k \right\| .$$

The Poisson boundary of the operator P is trivial if and only if

$$\lim_{n\to\infty} \frac{1}{n+1} \left\| \sum_{k=0}^{n} (\theta - \theta')P^k \right\| = 0$$

for any two probability measures $\theta, \theta' \prec m$.

COROLLARY: *For a measure θ on X let $[\theta\colon P]$ be the minimal measure type dominating all the measures θP^n, $n \geq 0$, i.e., the measure type of linear combinations $\sum \alpha_k \theta P^k$, $\alpha_k > 0\ \forall\, k \geq 0$. Then the harmonic measures ν_θ and $\nu_{\theta'}$ are mutually singular if and only if the measure types $[\theta\colon P]$ and $[\theta'\colon P]$ are.*

1.2.6 As it follows from the isomorphism $H^\infty(X, m, P) \cong L^\infty(\Gamma, \boldsymbol{\nu})$, the Poisson boundary is finite if and only if the space of bounded harmonic functions is finite-dimensional.

We shall say that the Poisson boundary Γ of a Markov operator P is **purely non-atomic** if the harmonic measure class $\boldsymbol{\nu}$ has no atoms, and that it is **purely atomic** if the measure class $\boldsymbol{\nu}$ has no continuous part. Markov operators with atoms in the Poisson boundary can be characterized in the following way.

A positive harmonic function f of the operator P is called **minimal** if all positive harmonic functions dominated by f are multiples of f, i.e.,

$$0 \le f' = Pf' \le f \qquad \Longleftrightarrow \qquad \exists C \in [0,1]\colon f' = Cf \,.$$

In particular, triviality of the Poisson boundary (absence of non-constant bounded harmonic functions) means that the constant harmonic function $\mathbf{1}$ is minimal.

Since the isomorphism $H^\infty(X, m, P) \cong L^\infty(\Gamma, \boldsymbol{\nu})$ preserves positivity, minimal bounded harmonic functions are in one-to-one correspondence with minimal functions from the space $L^\infty(\Gamma, \boldsymbol{\nu})$, i.e., with atoms of the measure type $\boldsymbol{\nu}$. This correspondence is explicitly given by the formula:

$$\varphi_\gamma(x) = \nu_x(\gamma) \,.$$

Thus, we have

PROPOSITION: *The Poisson boundary of a Markov operator P: $L^\infty(X,m)\hookleftarrow$ is purely non-atomic if and only if there are no bounded minimal P-harmonic functions.*

1.3 THE POISON KERNEL

1.3.1 Definition: Let P: $L^\infty(X,m)\hookleftarrow$ be a Markov operator, and $\nu \sim \boldsymbol{\nu}$ – a reference probability measure on its Poisson boundary $\Gamma = \Gamma(P)$. A measurable non-negative kernel $\Pi = \Pi_\nu$ on the product $X \times \Gamma$ is called a **Poisson kernel** if for any probability measure $\theta \prec m$

$$\frac{d\nu_\theta}{d\nu}(\gamma) = \int \Pi(x,\gamma)\, d\theta(x) \,.$$

1.3.2 Existence of a Poisson kernel is equivalent to absolute continuity of m-almost all harmonic measures ν_x with respect to the harmonic measure type $\boldsymbol{\nu}$, so that

$$\Pi(x,\gamma) = \frac{d\nu_x}{d\nu}(\gamma) \,,$$

and the Poisson formula takes the form

$$f(x) = \langle \nu_x, f \rangle = \int \widehat{f}(\gamma)\, \Pi(x,\gamma)\, d\nu(\gamma) \,.$$

The Poisson kernel always exists if the operator P has transition densities $p(\cdot,\cdot)$ with respect to the measure m. In this case $\nu_x = \nu_\theta \prec \boldsymbol{\nu}$ for the measure $\theta = \delta_x P = p(x,\cdot)m \prec m$.

1.3.3 The functions on X

$$\varphi_\gamma(x) = \Pi(x,\gamma) = \frac{d\nu_x}{d\nu}(\gamma)$$

are for ν-a.e. point $\gamma \in \Gamma$ minimal (unbounded, unless γ is an atom) P-harmonic functions, hence the Poisson formula gives a representation of a bounded harmonic function as an integral of minimal ones. Note that (almost all) minimal harmonic functions φ_γ belong to extreme rays of a simplex, so that if a positive harmonic function admits a representation as an integral of these functions with respect to a certain representing measure, then this representation is unique [Dy].

1.3.4 For an initial probability distribution $\theta \prec m$ the formula

$$_\theta\mathbf{P}^{\mathbb{Z}_+} = \int {_\theta\mathbf{P}_\gamma^{\mathbb{Z}_+}}\, d\nu_\theta(\gamma)$$

gives the **conditional decomposition** of the measure $_\theta\mathbf{P}^{\mathbb{Z}_+}$ with respect to the Poisson boundary (i.e., its **ergodic decomposition** with respect to the shift in the path space). The **conditional measure** $_\theta\mathbf{P}_\gamma^{\mathbb{Z}_+}$ is a Markov measure in the path space of the Markov operator

$$P_\gamma = \mathbf{M}_\gamma^{-1} P \mathbf{M}_\gamma \ ,$$

which is called the **Doob transform** of the original operator P (here $\mathbf{M}_\gamma$ is the operator of multiplication by φ_γ). The initial distribution θ_γ of the measure $_\theta\mathbf{P}_\gamma^{\mathbb{Z}_+}$ is determined by the relation

$$\frac{d\theta_\gamma}{d\theta}(x) = \frac{d\nu_x}{d\nu_\theta}(\gamma)\ .$$

1.3.5 Below alongside with the condition

(P) The Markov operator $P\colon L^\infty(X,m)\hookleftarrow$ has a Poisson kernel.

we shall often use a stronger condition

(P′) Almost all harmonic measures ν_x, $x \in X$ of the operator $P\colon L^\infty(X,m)\hookleftarrow$ are equivalent to the harmonic measure type $\boldsymbol{\nu}$.

Condition (P′) means that the Poisson kernel is almost everywhere positive. Note that condition (P′) automatically implies the irreducibility condition (Irr), but the converse is not true.

1.4 The Poisson extension

1.4.1 Definition: Let $P\colon L^\infty(X,m)\hookleftarrow$ be a Markov operator satisfying condition (Stat). Let $\Gamma = \Gamma(P)$ be its Poisson boundary, and let $X^\infty = X \times \Gamma$. The image of the measure ${}_m\mathbf{P}^{\mathbb{Z}}$ under the map

$$\overline{x} = \{x_n\} \mapsto \{(x_n, \mathbf{bnd}\overline{x})\}$$

is a shift-invariant Markov measure on the space $(X^\infty)^{\mathbb{Z}}$ with the one-dimensional distribution

$$dm^\infty(x,\gamma) = dm(x)\, d\nu_x(\gamma),$$

which determines a Markov operator $P^\infty\colon L^\infty(X^\infty, m^\infty)\hookleftarrow$ on the space X^∞ with stationary measure m^∞. The operator P^∞ is called the **Poisson extension** of the operator P.

Remark: In fact, this notion (in an implicit form) was used by Sullivan [Su] in his proof of the equivalence of ergodicity of the geodesic flow on a Riemannian manifold of constant negative curvature and recurrence of the Brownian motion on this manifold (see also [K10]).

1.4.2 The operator P^∞ and its reversed operator $\check{P}^\infty$ have the following interpretation: on almost every cross-section $X \times \{\gamma\}$ the operator P^∞ coincides with the conditional operator P_γ, and the operator $\check{P}^\infty$ coincides with the reversed operator $\check{P}$. In other words, if $f_\gamma(x) = f(x,\gamma)$, then

$$P^\infty f(x,\gamma) = P_\gamma f_\gamma(x)\ ,$$

and

$$\check{P}^\infty f(x,\gamma) = \check{P} f_\gamma(x)\ .$$

1.4.3 Let $\mathbf{bnd}\check{}$ be the map assigning to a path $\overline{x}$ from the bilateral path space $X^{\mathbb{Z}}$ the corresponding point from the Poisson boundary $\check{\Gamma}$ of the reversed operator $\check{P}$. Denote by $\check{\boldsymbol{\nu}}$ the harmonic measure type on the Poisson boundary $\check{\Gamma}$, and by $\boldsymbol{\nu}^\pm$ the measure type on the product $\check{\Gamma} \times \Gamma$ induced by the map $\overline{x} \mapsto (\mathbf{bnd}\check{}\,\overline{x}, \mathbf{bnd}\overline{x})$ from the bilateral path space $(X^{\mathbb{Z}}, {}_m\mathbf{P}^{\mathbb{Z}})$. [Warning: the measure type $\boldsymbol{\nu}^\pm$ in general is *not* the product of the measure types $\check{\boldsymbol{\nu}}$ and $\boldsymbol{\nu}$.]

THEOREM: *Let P: $L^\infty(X,m)\hookleftarrow$ be a Markov operator satisfying condition* (Stat). *Then the Poisson boundary of its Poisson extension P^∞ coincides with the Poisson boundary of the operator P, and the Poisson boundary of the reversed operator $\check{P}^\infty$ is the product $\check{\Gamma}\times\Gamma$ of the Poisson boundaries of the operators $\check{P}$ and P with the harmonic measure type $\boldsymbol{\nu}^\pm$. If, in addition, the operator P satisfies condition* (P′), *then the measure type $\boldsymbol{\nu}^\pm$ is the product of measure types $\check{\boldsymbol{\nu}}$ and $\boldsymbol{\nu}$.*

Proof: The description of the Poisson boundaries of the operators P^∞ and $\check{P}^\infty$ immediately follows from the structure of the (common) bilateral path space of the operators P^∞ and $\check{P}^\infty$.

If the operator P satisfies condition (P), then for a given reference probability measure $\nu\sim\boldsymbol{\nu}$ the measure m^∞ can be presented as

$$dm^\infty(x,\gamma)=dm(x)\,d\nu_x(\gamma)=\frac{d\nu_x}{d\nu}(\gamma)\,dm(x)\,d\nu(\gamma)\,.$$

If P satisfies stronger condition (P′), then the conditional measure of the measure m^∞ on almost every cross-section $X\times\{\gamma\}$ is equivalent to m. Since on the leaf $X\times\{\gamma\}$ the operator $\check{P}^\infty$ coincides with the reversed operator $\check{P}$, we obtain that for ν-a.e. $\gamma\in\Gamma$ conditioning by γ gives the measure type $\check{\boldsymbol{\nu}}$ on the cross-section $\check{\Gamma}\times\{\gamma\}\subset\check{\Gamma}\times\Gamma$. After integrating with respect to the distribution of γ (which belongs to the measure type $\boldsymbol{\nu}$), we obtain that the measure type $\boldsymbol{\nu}^\pm$ is the product of the measure types $\check{\boldsymbol{\nu}}$ and $\boldsymbol{\nu}$. ∎

1.4.4 Below we shall also need the following technical result.

PROPOSITION: *Let P: $L^\infty(X,m)\hookleftarrow$ be a Markov operator satisfying condition* (Stat), *and let $\{\mu_x\}$, $x\in X$ be a family of probability measures on the Poisson boundary Γ of the operator P absolutely continuous with respect to the harmonic measure type $\boldsymbol{\nu}$. Then the measure $d\mu(x,\gamma)=dm(x)\,d\mu_x(\gamma)$ is a stationary measure of the operator $\check{P}^\infty$ if and only if the densities $d\mu_x/d\nu(\gamma)$ are P-harmonic functions on X for* a.e. *$\gamma\in\Gamma$, where $\nu\sim\boldsymbol{\nu}$ is a reference probability measure on Γ.*

Proof: The measure μ can be presented as

$$d\mu(x,\gamma)=dm(x)\,d\mu_x(\gamma)=\frac{d\mu_x}{d\nu}(\gamma)\,dm(x)\,d\nu(\gamma)\,.$$

Thus, on a.e. cross-section $X\times\{\gamma\}$ it has the density $d\mu_x/d\nu(\gamma)$ with respect to the measure m. Since the operator $\check{P}^\infty$ coincides with the operator $\check{P}$ on the leaf $X \times \{\gamma\}$, stationarity of the measure μ with respect to the operator $\check{P}^\infty$ is equivalent to stationarity of almost all measures $d\mu_x/d\nu(\gamma)dm(x)$ with respect to the operator $\check{P}$, the latter being equivalent to P-harmonicity of the densities $d\mu_x/d\nu(\gamma)$ by Proposition 1.1.8. ∎

2. Covering Markov operators

In this Section we shall introduce the principal object of our study — the covering Markov operator, which is a Markov operator on a covering measure space invariant with respect to its deck transformations group. We state here some basic properties of the Poisson boundary of covering Markov operators. We give an elementary proof of the fact that the Poisson boundary of a covering operator with a non-amenable deck group is always infinite (§2.2) and show that the deck group action on the Poisson boundary is always amenable (§2.3). In §2.4 we consider corecurrent covering operators. The main technical tool used below is Theorem 2.4.5 which shows that the Poisson extension of a corecurrent operator is also corecurrent. It implies that for a corecurrent covering operator $\widetilde{P}$ the action of the deck group on the product of Poisson boundaries of the operator $\widetilde{P}$ and its reversed operator is ergodic (Theorem 2.4.6).

2.1 Covering spaces and operators.

2.1.1 Let $(\widetilde{X}, \widetilde{m})$ be a measure space with a measure preserving **completely dissipative** action of a countable group G. It means that there is a set $X^0 \subset \widetilde{X}$ such that all its translations gX^0, $g \in G$ are pairwise disjoint, and their union is the whole space $\widetilde{X}$ (mod 0). Then one can identify the quotient space $X = \widetilde{X}/G$ with the "fundamental domain" X^0. For a point $x \in \widetilde{X}$ let $g(x) \in G$ be defined by the relation $x \in g(x)X^0$, and let $\pi(x) = g(x)^{-1}x \in X^0$ be the **projection** from $\widetilde{X}$ onto $X^0 \cong X$. Then the points $x \in X$ can be identified with pairs $\big(g(x), \pi(x)\big) \in G \times X^0$. Interrelations between the spaces $\widetilde{X}, X$ and X^0 are illustrated by the following diagram:

$$X^0 \hookrightarrow \widetilde{X} \cong G \times X^0 \xrightarrow{\pi} X \cong X^0 .$$

By m^0 denote the restriction of the measure $\widetilde{m}$ onto X^0, and by m the image of m^0 with respect to the projection $\widetilde{X} \to X$. We shall say that $(\widetilde{X}, \widetilde{m})$ is a **covering measure space** with the deck transformations group G, and (X, m) is its **quotient space**.

2.1.2 Definition: A Markov operator $\widetilde{P}$: $L^\infty(\widetilde{X},\widetilde{m})\hookleftarrow$ on a covering measure space $(\widetilde{X},\widetilde{m})$ is a **covering Markov operator** if it commutes with the action of the deck group G, i.e.,

$$\widetilde{P}(gf) = g(\widetilde{P}f) \qquad \forall f \in L^\infty(\widetilde{X},\widetilde{m}) .$$

Thus, $\widetilde{P}$ acts on the subspace of G-invariant functions in $L^\infty(\widetilde{X},\widetilde{m})$, i.e., $\widetilde{P}$ determines a **quotient Markov operator** P: $L^\infty(X,m)\hookleftarrow$.

2.1.3 In the case when the operator $\widetilde{P}$ has transition densities $\widetilde{p}(\cdot,\cdot)$ with respect to the measure $\widetilde{m}$, transition densities of the quotient operator P are given by the formula

$$p(x,y) = \sum_{g\in G} \widetilde{p}(\widetilde{x}, g\widetilde{y}) ,$$

where $x = \pi(\widetilde{x})$ and $y = \pi(\widetilde{y})$.

One can easily verify that m is a stationary measure of the quotient operator P if and only if $\widetilde{m}$ is a stationary measure of the covering operator $\widetilde{P}$.

2.1.4 If $\widetilde{P}$: $L^\infty(\widetilde{X},\widetilde{m})\hookleftarrow$ is a covering Markov operator, then the deck group G acts on the Poisson boundary $\Gamma = \Gamma(\widetilde{P})$ (because the coordinate-wise action of G on the path space commutes with the time shift), and

$$\nu_{g\theta} = g\nu_\theta \qquad \forall \theta \prec \widetilde{m} ,$$

so that this action preserves the harmonic measure type $\boldsymbol{\nu}$ on Γ. The space of bounded G-invariant functions on $\Gamma(\widetilde{P})$ is isomorphic to the space of bounded G-invariant $\widetilde{P}$-harmonic functions on $\widetilde{X}$, i.e., to the space of bounded P-harmonic functions on X, the latter being isomorphic to the space of bounded measurable functions on the Poisson boundary $\Gamma(P)$. Thus, we have

THEOREM: *The space of G-ergodic components of the Poisson boundary $\Gamma(\widetilde{P})$ of a covering Markov operator $\widetilde{P}$: $L^\infty(\widetilde{X},\widetilde{m})\hookleftarrow$ is isomorphic to the Poisson boundary of the quotient operator P. In particular, the action of the deck group G on $\Gamma(\widetilde{P})$ is ergodic if and only if the operator P is Liouville.*

This theorem permits us to consider in the sequel without any loss of generality only the case when the quotient Markov operator P has the Liouville property, i.e., its Poisson boundary is trivial. In particular, the quotient operator P can be always assumed to satisfy condition (Irr).

2.1.5 More generally, if $H \subset G$ is a normal subgroup of the deck group G, then the space of ergodic components Γ^H in Γ with respect to the action of H admits the following interpretation.

Factorization of the space $(\widetilde{X}, \widetilde{m})$ by the group H gives a covering measure space $(\widetilde{X}^H, \widetilde{m}^H)$ with the same base space (X, m) and the deck group G/H. The covering operator $\widetilde{P}$ preserves the subspace of H-invariant functions in $L^\infty(\widetilde{X}, \widetilde{m})$, which means that it defines a covering operator $\widetilde{P}^H$: $L^\infty(\widetilde{X}^H, \widetilde{m}^H)\hookleftarrow$ with the deck group G/H and the same quotient P: $L^\infty(X, m)\hookleftarrow$. Since the operator $\widetilde{P}$ can be considered as a covering operator of the operator $\widetilde{P}^H$ with the deck group H, Theorem 2.1.4 implies

THEOREM: *If $\widetilde{P}$: $L^\infty(\widetilde{X}, \widetilde{m})\hookleftarrow$ is a covering Markov operator, and $H \subset G$ is a normal subgroup of the deck group G, then the Poisson boundary of the operator $\widetilde{P}^H$: $L^\infty(\widetilde{X}^H, \widetilde{m}^H)\hookleftarrow$ coincides with the space of ergodic components Γ^H of the Poisson boundary Γ of the operator $\widetilde{P}$ with respect to the action of H.*

2.2 Non-amenable covers.

2.2.1 Recall that a countable group G is called **amenable** if there exists a finitely additive, translation invariant probability measure defined for all subsets of G. According to **Reuter's criterion**, amenability of G is equivalent to existence of a sequence λ_n of (σ-additive) probability measures which strongly converges to a (left-invariant) mean on G, i.e.,

$$\|g\lambda_n - \lambda_n\| \to 0 \qquad \forall g \in G$$

(these are just two from a very long list of equivalent definitions – see [Gr], [Paa], [Pi]).

2.2.2 The following result has been known for long time for random walk on groups (e.g., see [Fu3], [KV]). For covering operators corresponding to the Brownian motion on Riemannian manifolds it was proved in [LS] by using a projection from the space of all bounded measurable functions on the covering space to the space of bounded harmonic functions. Our proof uses a more direct approach (a simplified version of that of [KV]).

THEOREM: *Any covering Markov operator with a non-amenable deck transformations group has a non-trivial Poisson boundary.*

Proof: Suppose that the Poisson boundary Γ of the operator $\widetilde{P}$ is trivial. Take a probability measure $\theta \prec m$. Then by Proposition 1.2.5 applied to the measures θ and $g\theta$, $g \in G$

$$\frac{1}{n+1}\left\|\sum_{k=0}^{n}(\theta - g\theta)P^k\right\| \underset{n\to\infty}{\longrightarrow} 0 \qquad \forall g \in G\,.$$

In other words, if

$$\theta_n = \frac{1}{n+1}\sum_{k=0}^{n}\theta P^k\,,$$

then

$$\|\theta_n - g\theta_n\| \to 0 \qquad \forall g \in G\,.$$

Denote by λ_n the probability measure on G which is the image of the measure θ_n under the map $x \mapsto g(x)$ with respect to a fixed fundamental domain X^0. Then

$$\|\lambda_n - g\lambda_n\| \le \|\theta_n - g\theta_n\| \qquad \forall g \in G\,,$$

so that G must be amenable by Reuter's criterion. ∎

Theorem 2.2.2 implies the following two generalisations.

2.2.3 THEOREM: *The Poisson boundary of any covering Markov operator with a non-amenable deck transformations group is infinite.*

Proof: Suppose that the Poisson boundary Γ of the operator $\widetilde{P}$ is finite and consists of n atoms. The group G acts on Γ, hence we have a homomorphism $\psi\colon G \to \mathfrak{S}_n$ of the group G to the symmetric group $\mathfrak{S}_n$. Let $G_0 = \ker\psi$. Then the group G_0 is also non-amenable, because it has a finite index in G.

Consider the conditional operator $\widetilde{P}_\gamma$ corresponding to a point $\gamma \in \Gamma$. Then its Poisson boundary is trivial. On the other hand, the operator $\widetilde{P}_\gamma$ is G_0-invariant, and can be considered as a covering operator with the deck transformations group G_0. Thus, triviality of the Poisson boundary of the operator $\widetilde{P}_\gamma$ contradicts to non-amenability of G_0 by Theorem 2.2.2. ∎

Remark: For covering Markov operators corresponding to the Brownian motion on regular covers of compact Riemannian manifolds Theorem 2.2.3 was earlier proved by Kifer [Ki] (in a rather complicated way) and Toledo [To]. Using the notion of the Poisson boundary makes this statement much easier to prove. Note that for regular covers of compact Riemannian manifolds this result readily follows from the entropy theory [K4]: in this case the Poisson boundary is either trivial or purely non-atomic (for, if it is non-trivial then almost all extreme harmonic functions grow exponentially along paths of the corresponding conditional process; also see below Corollary 1 of Theorem 3.3.1).

2.2.4 THEOREM: *The stabilizer subgroup*

$$\mathrm{Stab}_\gamma = \{g \in G\colon g\gamma = \gamma\} \subset G$$

is amenable for almost all (with respect to the harmonic measure type) points of the Poisson boundary of any covering Markov operator.

Proof: It uses the same idea as in the proof of Theorem 2.2.3. Let $\widetilde{P}_\gamma$ be the conditional operator coresponding to a point γ. Then it is Stab_γ-invariant, so that it can be considered as a covering operator with the deck group Stab_γ. Its Poisson boundary is trivial, hence Stab_γ must be amenable by Theorem 2.2.2. ∎

2.3 The Poisson boundary as a Mackey range.

2.3.1 Let $\widetilde{P}\colon L^\infty(\widetilde{X},\widetilde{m})\hookleftarrow$ be a covering Markov operator satisfying condition (Stat), and let $(\widetilde{X}^{\mathbb{Z}}, {}_{\widetilde{m}}\mathbf{P}^{\mathbb{Z}})$ be the corresponding bilateral path space. The space $(\widetilde{X}^{\mathbb{Z}}, {}_{\widetilde{m}}\mathbf{P}^{\mathbb{Z}})$ is endowed with the measure preserving coordinate-wise action of the group G and is completely dissipative with respect to this action, so that it is a covering space. Its quotient space $(\widetilde{X}^{\mathbb{Z}}, {}_{\widetilde{m}}\mathbf{P}^{\mathbb{Z}})/G$ can be described in the following way.

Denote by $h_n = g(x_{n-1})^{-1}g(x_n)$ the **increments** of the sequence $g(x_n)$, where $\{x_n\} = \overline{x} \in \widetilde{X}^{\mathbb{Z}}$. Let φ be the map from the path space $\widetilde{X}^{\mathbb{Z}}$ to the space $(X\times G)^{\mathbb{Z}}$ of $X \times G$-valued sequences defined as

$$[\varphi(\overline{x})]_n = \big(\pi(x_n), h_n\big) \in X \times G \qquad n \in \mathbb{Z}\,,$$

then

$$\varphi(g\overline{x}) = \varphi(\overline{x}) \qquad \forall \overline{x} \in \widetilde{X},\ g \in G\,,$$

and the map

$$\Phi(\overline{x}) = \big(\varphi(x), g(x_0)\big)$$

is a one-to-one correspondence between the spaces $\widetilde{X}^{\mathbb{Z}}$ and $(X \times G)^{\mathbb{Z}} \times G$. In other words, the set $\{\overline{x} \in \widetilde{X}^{\mathbb{Z}}\colon g(x_0) = e\} = A^0$ is a fundamental domain for the

G-action in the path space $\widetilde{X}^{\mathbb{Z}}$. Thus, if λ is the image under the map φ of the restriction of the measure ${}_{\widetilde{m}}\mathbf{P}^{\mathbb{Z}}$ onto A^0, then the measure space $\big((X\times G)^{\mathbb{Z}}, \lambda\big)$ can be identified with the quotient space of the covering measure space $(\widetilde{X}^{\mathbb{Z}}, {}_m\mathbf{P}^{\mathbb{Z}})$.

2.3.2 The map φ intertwines the shift T in the space $\widetilde{X}^{\mathbb{Z}}$ and the shift S in the space $(X \times G)^{\mathbb{Z}}$, so that

$$S\varphi(\overline{x}) = \varphi(T\overline{x}) \qquad \forall \overline{x} \in \widetilde{X}^{\mathbb{Z}} .$$

The $\mathbb{Z}$-action $\{T^n\}$ on $\widetilde{X}^{\mathbb{Z}}$ commutes with the G-action, and the measure ${}_{\widetilde{m}}\mathbf{P}^{\mathbb{Z}}$ is T-invariant, hence the measure λ is S-invariant. Since

$$\Phi(T\overline{x}) = \big(\varphi(T\overline{x}), g((T\overline{x})_0)\big) = \big(S\varphi(\overline{x}), g(x_1)\big) = \big(S\varphi(\overline{x}), g(x_0)h_1\big) ,$$

the $\mathbb{Z}$-action $\{T^n\}$ on the space $(\widetilde{X}^{\mathbb{Z}}, {}_m\mathbf{P}^{\mathbb{Z}})$ is the **skew product** of the $\mathbb{Z}$-action $\{S^n\}$ on the space $\big((X \times G)^{\mathbb{Z}}, \lambda\big)$ and the G-valued **cocycle** α of the group $\mathbb{Z}$ determined by the function $f_\alpha = \alpha(1,\cdot)$ on $(X \times G)^{\mathbb{Z}}$:

$$f_\alpha\big(\{\pi(x_n), h_n\}\big) = h_1 .$$

2.3.3 In the same way the semigroup $\mathbb{Z}_+$-action of the shift T^+ in the unilateral path space $(\widetilde{X}^{\mathbb{Z}_+}, {}_{\widetilde{m}}\mathbf{P}^{\mathbb{Z}_+})$ is the skew product of the $\mathbb{Z}_+$-action by unilateral shifts in the space $\big((X\times G)^{\mathbb{Z}_+}, \lambda_+\big)$ and the corresponding cocycle α_+ of the semigroup $\mathbb{Z}_+$ (here λ_+ is the projection of the measure λ). By definition, the Poisson boundary $\Gamma(\widetilde{P})$ of the operator $\widetilde{P}$ is the space of $\mathbb{Z}_+$-ergodic components in $(\widetilde{X}^{\mathbb{Z}}, {}_m\mathbf{P}^{\mathbb{Z}_+})$, or, in other terms, the **Mackey range** of the cocycle α_+ [Z2], [Sch]. Theorem 3.3 from [Z1] and its modification for $\mathbb{Z}_+$-cocycles (*ibid*, Theorem 5.2) now imply

THEOREM: *If* $\widetilde{P}$: $L^\infty(\widetilde{X}, \widetilde{m})\hookleftarrow$ *is a covering Markov operator satisfying condition* (Stat), *then the action of the deck group* G *on the Poisson boundary of the operator* $\widetilde{P}$ *is amenable.*

2.4 CORECURRENT OPERATORS.

2.4.1 Definition: A Markov operator P: $L^\infty(X,m)\hookleftarrow$ is called **recurrent** (or, **conservative**) if it satisfies the following equivalent conditions:

(Rec$_1$) If $f \in L^\infty(X,m)$, and $Pf \le f$ (i.e., f is **superharmonic**), then $Pf = f$, and for any $t \ge 0$ the set $A_t = \{x \in X\colon f(x) \ge t\}$ is P-invariant;

(Rec$_2$) The shift T^+ in the unilateral path space $(X^{\mathbb{Z}_+}, {}_m\mathbf{P}^{\mathbb{Z}_+})$ is **conservative**;

(Rec$_3$) Any set $A \subset X$ with $m(A) > 0$ is **recurrent**, i.e., it is visited infinitely often by ${}_m\mathbf{P}^{\mathbb{Z}_+}$-a.e. unilateral path starting from A.

In particular, if a Markov operator P: $L^\infty(X, m) \hookleftarrow$ is irreducible, then its recurrence is equivalent to absence of non-constant bounded superharmonic functions on X, so that the Poisson boundary of any irreducible recurrent operator is trivial. Another property equivalent to recurrence for irreducible operators is that $\sum_n P^n f \equiv \infty$ for any non-negative function f which is not identically zero [Fo], [Kr].

2.4.2 Proposition: *A Markov operator P: $L^\infty(X, m) \hookleftarrow$ with stationary measure m is irreducible and recurrent if and only if its reversed operator $\check{P}$ is.*

Proof: Let $A \subset X$ be an invariant set of the operator P. Then the function $\mathbf{1}_A$ is constant along ${}_m\mathbf{P}^{\mathbb{Z}_+}$-a.e. unilateral path from $X^{\mathbb{Z}_+}$. Since the measure ${}_m\mathbf{P}^{\mathbb{Z}}$ is the bilateral extension of the measure ${}_m\mathbf{P}^{\mathbb{Z}_+}$, the same is true for ${}_m\mathbf{P}^{\mathbb{Z}}$-a.e. bilateral path from $X^{\mathbb{Z}}$, so that A is an invariant set of the reversed operator $\check{P}$. Conversely, any $\check{P}$-invariant set is P-invariant. Thus, P and $\check{P}$ have the same invariant sets and, in particular, they are irreducible simultaneously.

Further, conservativity of the unilateral shift T^+ in the path space $(X^{\mathbb{Z}_+}, {}_m\mathbf{P}^{\mathbb{Z}_+})$ is equivalent to conservativity of the bilateral shift T in the space $(X^{\mathbb{Z}}, {}_m\mathbf{P}^{\mathbb{Z}})$. Therefore, conservativity of T^+ is equivalent to conservativity of the unilateral shift T^- corresponding to the reversed operator $\check{P}$. ∎

2.4.3 We shall say that a covering Markov operator $\widetilde{P}$: $L^\infty(\widetilde{X}, \widetilde{m}) \hookleftarrow$ is **corecurrent** if the quotient operator P: $L^\infty(X, m) \hookleftarrow$ is recurrent.

Recall that a Markov operator P: $L^\infty(X, m) \hookleftarrow$ is **Harris recurrent** [Fo], [Kr] if it is irreducible, recurrent and there exists $n > 0$ such that P^n dominates a non-trivial integral kernel, i.e., the set of points $x \in X$ such that the n-step transition probabilities π_x^n are non-singular with respect to the measure m is not negligible. A Harris recurrent operator has a unique (up to a multiplier) σ-finite stationary measure equivalent to m.

Let Q_n be the absolutely continuous part of the power P^n (i.e., Q_n is the maximal integral kernel dominated by P^n). If P is a Harris operator, then $Q_n\mathbf{1} \uparrow \mathbf{1}$. Thus, for m-a.e. point $x \in X$ the total mass of the singular part of the transition probabilities π_x^n tends to zero. If P is the quotient of a covering

Markov operator $\widetilde{P}$: $L^\infty(\widetilde{X},\widetilde{m})\hookleftarrow$, then it implies that the same must be true for $\widetilde{P}$, i.e., for $\widetilde{m}$-a.e. point $x \in \widetilde{X}$ the total mass of the singular part of the transition probabilities $\widetilde{\pi}_x^n$ tends to zero. Thus, for $\widetilde{m}$-a.e. point $x \in \widetilde{X}$ the harmonic measure ν_x on the Poisson boundary of the operator $\widetilde{P}$ is absolutely continuous with respect to the harmonic measure type $\boldsymbol{\nu}$. So, we have

PROPOSITION: *If $\widetilde{P}$: $L^\infty(\widetilde{X},\widetilde{m})\hookleftarrow$ is a covering Markov operator such that its quotient is Harris recurrent, then the operator $\widetilde{P}$ satisfies condition* (P).

2.4.4 Suppose that $\widetilde{m}$ is a stationary measure of a covering Markov operator $\widetilde{P}$: $L^\infty(\widetilde{X},\widetilde{m})\hookleftarrow$ (i.e., m is a stationary measure of the quotient operator P). Let $\widetilde{X}^\infty = \widetilde{X} \times \Gamma(\widetilde{P})$, and let $\widetilde{P}^\infty$: $L^\infty(\widetilde{X}^\infty,\widetilde{m}^\infty)\hookleftarrow$ be the Poisson extension of the operator $\widetilde{P}$. The measure $d\widetilde{m}^\infty(x,\gamma) = d\widetilde{m}(x)\, d\nu_x(\gamma)$ is G-invariant, so that the space $(\widetilde{X}^\infty,\widetilde{m}^\infty)$ is a covering measure space with the deck group G. Let (X^∞, m^∞) be the corresponding quotient space. As a measure space, (X^∞, m^∞) is isomorphic to the product of the quotient space (X,m) and the Poisson boundary $\Gamma(\widetilde{P})$. The operator $\widetilde{P}^\infty$ is G-invariant, so that it determines a quotient Markov operator P^∞ on the space (X^∞, m^∞). [We are slightly abusing the notations: the operator P^∞ is *not* the Poisson extension of the quotient operator P.]

2.4.5 THEOREM: *If $\widetilde{P}$: $L^\infty(\widetilde{X},\widetilde{m})\hookleftarrow$ is a corecurrent covering Markov operator with an irreducible quotient, then the operator P^∞: $L^\infty(X^\infty,m^\infty)\hookleftarrow$ is irreducible and recurrent.*

Proof: First we shall show irreducibility of the operator P^∞. If there is a non-trivial P^∞-invariant subset $A^\infty \subset X^\infty$, then its pullback $\widetilde{A}^\infty \subset \widetilde{X}^\infty$ is $\widetilde{P}^\infty$-invariant and G-invariant simultaneously. Thus, the set of all paths which remain in $\widetilde{A}^\infty$ determines a non-trivial G-invariant subset of the Poisson boundary $\Gamma(\widetilde{P}^\infty) = \Gamma(\widetilde{P})$, or, in other words, a non-trivial subset of the Poisson boundary of the quotient operator P. On the other hand, since P is irreducible and recurrent, its Poisson boundary is trivial, which gives a contradiction.

Now, since P^∞ is irreducible, its recurrence would follow from existence of a finite measure recurrent set in its state space. Let A be a subset of X with $0 < m(A) < \infty$, and $\widetilde{A}$ its pullback to $\widetilde{X}$. Put $\widetilde{A}^\infty = \widetilde{A} \times \Gamma \subset \widetilde{X}^\infty$. The set $\widetilde{A}^\infty$ is G-invariant, so that $m^\infty(A^\infty) = m(A) < \infty$ for its image $A^\infty \subset X^\infty$. As it follows from recurrence of the operator P, the set A is recurrent, which implies that the set A^∞ is also recurrent. ∎

2.4.6 THEOREM: *Let $\widetilde{P}$: $L^\infty(\widetilde{X}, \widetilde{m})\hookleftarrow$ be a corecurrent covering Markov operator with an irreducible quotient. If the operator $\widetilde{P}$ satisfies condition (P′), then the action of the deck group G on the product of the Poisson boundaries of the operator $\widetilde{P}$ and the reversed operator $\check{\widetilde{P}}$ is ergodic with respect to the product of corresponding harmonic measure types.*

Proof: The proof uses the fact that the product of the Poisson boundaries of the operators $\widetilde{P}$ and $\check{\widetilde{P}}$ is the Poisson boundary of the reversed operator $\check{\widetilde{P}}^\infty$, and that under condition (P′) the harmonic measure type of the operator $\check{\widetilde{P}}^\infty$ is the product of harmonic measure types of the operators $\widetilde{P}$ and $\check{\widetilde{P}}$ (Proposition 1.4.3). The quotient $\check{P}^\infty$ of the operator $\check{\widetilde{P}}^\infty$ is irreducible and recurrent by Theorem 2.4.5, so that the action of G is ergodic by Theorem 2.1.4. ∎

COROLLARY: *If the operator $\widetilde{P}$ is reversible, then the action of the deck group G on the square of its Poisson boundary (with the product measure type) is ergodic.*

Remark: One can easily construct examples of cotransient operators $\widetilde{P}$ such that the G-action on the product of the Poisson boundaries of the operators $\widetilde{P}$ and $\check{\widetilde{P}}$ is still ergodic (any transient operator P such that both P and $\check{P}$ are Liouville provides a trivial example of this kind). However, these two properties (corecurrence of $\widetilde{P}$ and G-ergodicity of the product $\Gamma(\widetilde{P}) \times \Gamma(\check{\widetilde{P}})$) turn out to be equivalent for a class of Markov operators on **hyperbolic spaces** [K10].

3. Rigidity of Radon–Nikodym cocycles on the Poisson boundary

In this Section we consider the cohomological properties of the Radon–Nikodym cocycles on the Poisson boundary of corecurrent Markov operators, and prove the following rigidity type result: the cohomology class of the Radon–Nikodym derivatives of a G-invariant quotient of the Poisson boundary determines this quotient (Theorem 3.2.1). As a corollary we obtain that conditional measures corresponding to any two distinct G-invariant partitions $\xi \prec \zeta$ of the Poisson boundary are purely non-atomic (Theorem 3.3.1). Thus, the Poisson boundary is either trivial or purely non-atomic, and the action of any normal subgroup on the Poisson boundary is conservative (Theorem 3.3.3). In particular, any finite normal subgroup of the deck group acts trivially on the Poisson boundary.

3.1 MEASURABLE PARTITIONS OF G-SPACES.

3.1.1 We shall start with recalling some basic definitions from the theory of measurable partitions of **Lebesgue spaces** [CFS], [Ro].

Let (Ω, λ) be a probability Lebesgue space, and ξ — its measurable partition. We shall denote by

$$(\Omega^\xi, \lambda^\xi) = (\Omega, \lambda)/\xi$$

the corresponding **quotient** probability space which is the image of Ω under the map $\omega \mapsto \omega^\xi$ assigning to a point $\omega \in \Omega$ its equivalence class ω^ξ.

For objects connected with the quotient space Ω^ξ we shall use the same notations as for Ω with adding the superscript ξ. By λ^{ω^ξ} we shall denote the conditional measure of the measure λ with respect to the partition ξ conditioned at the point ω^ξ, so that

$$d\lambda(\omega) = d\lambda^\xi(\omega^\xi)\, d\lambda^{\omega^\xi}(\omega) \;,$$

or

$$\lambda = \int \lambda^{\omega^\xi}\, d\lambda^\xi(\omega^\xi)$$

(the **conditional decomposition** of the measure λ with respect to the partition ξ).

3.1.2 If μ is another probability measure absolutely continuous with respect to λ, then

$$\frac{d\mu}{d\lambda}(\omega) = \frac{d\mu^\xi}{d\lambda^\xi}(\omega^\xi)\, \frac{d\mu^{\omega^\xi}}{d\lambda^{\omega^\xi}}(\omega) \;, \tag{1}$$

which after integrating by the conditional measure λ^{ω^ξ} gives that almost surely

$$\frac{d\mu^\xi}{d\lambda^\xi}(\omega^\xi) = \int \frac{d\mu}{d\lambda}(\omega)\, d\lambda^{\omega^\xi}(\omega) \;.$$

In particular,

$$\frac{d\mu^\xi}{d\lambda^\xi}(\omega^\xi) = 0 \quad\Longleftrightarrow\quad \frac{d\mu}{d\lambda}(\omega) = 0 \qquad \lambda^{\omega^\xi}\text{-a.e. } \omega \in \Omega \;. \tag{2}$$

3.1.3 Suppose now that the space (Ω, λ) is endowed with a measure type preserving action of a countable group G. A measurable partition ξ of this space is called G-**invariant** if almost surely

$$\omega \overset{\xi}{\sim} \omega' \iff g\omega \overset{\xi}{\sim} g\omega' \qquad \forall\, g \in G\,.$$

It means that the corresponding **quotient space** $(\Omega^\xi, \lambda^\xi)$ is endowed with a measure type preserving action of G, and the projection $\omega \mapsto \omega^\xi$ is G-equivariant.

A natural class of G-invariant partitions are partitions ξ_H into **ergodic components** of the action of a normal subgroup $H \subset G$ (if H is not normal then the partition ξ_H is not G-invariant in general). For simplicity of notations, we shall use for objects connected with these partitions the superscript H instead of ξ_H.

Note that for a given G-invariant partition ζ of a Lebesgue space (Ω, λ) all G-invariant partitions ξ such that $\xi \preccurlyeq \zeta$ (i.e., ζ is a **refinement** of ξ) are in one-to-one correspondence with G-invariant partitions of the space Ω^ζ.

3.1.4 Applying formula (1) to the measure $\mu = g\lambda$ gives the following relation between the Radon–Nikodym derivatives of the translations of the measures λ and λ^ξ for a G-invariant measurable partition ξ:

$$\frac{dg\lambda}{d\lambda}(\omega) = \frac{dg\lambda^\xi}{d\lambda^\xi}(\omega^\xi)\, \frac{dg\lambda^{g^{-1}\omega^\xi}}{d\lambda^{\omega^\xi}}(\omega)\,, \tag{3}$$

where $g\lambda^{g^{-1}\omega^\xi}$ is the translation by g of the conditional measure $\lambda^{g^{-1}\omega^\xi}$ corresponding to the point $g^{-1}\omega^\xi$.

3.1.5 A positive measurable function β on $G \times \Omega$ is called a (multiplicative) **cocycle** [Sch], [Z2] if almost surely

$$\beta(g_1 g_2, \omega) = \beta(g_1, g_2\omega)\, \beta(g_2, \omega) \qquad \forall\, g_1, g_2 \in G\,.$$

Two cocycles β and β' are **cohomologous** (equivalent) if there exists a positive measurable function φ on Ω such that

$$\beta'(g, \omega) = \frac{\varphi(g\omega)}{\varphi(\omega)}\, \beta(g, \omega)\,.$$

If this function φ is measurable with respect to a partition ξ, then we shall say that β and β' are cohomologous over the corresponding quotient space Ω^ξ.

3.1.6 If ξ is a G-invariant measurable partition of the space (Ω, λ), then the cocycle

$$\Delta^{\xi}(g,\omega) = \frac{dg^{-1}\lambda^{\xi}}{d\lambda^{\xi}}(\omega^{\xi})$$

is called the **Radon–Nikodym cocycle** of the measure λ^{ξ} (it is convenient to consider the Radon–Nikodym derivatives as functions on Ω rather than on the quotient space Ω^{ξ}). By Δ we shall denote the Radon–Nikodym cocycle of the measure λ itself. Since Radon–Nikodym cocycles of equivalent measures are equivalent, one can speak about the cohomology classes of the measure types $[\lambda^{\xi}]$.

3.2 RADON–NIKODYM COCYCLES OF THE POISSON BOUNDARY.

3.2.1 Let now $\widetilde{P}$: $L^{\infty}(\widetilde{X}, \widetilde{m})\hookleftarrow$ be a covering Markov operator with the deck group G. The Poisson boundary Γ of the operator $\widetilde{P}$ is endowed with a natural G-action which preserves the harmonic measure type $\boldsymbol{\nu}$. We shall fix a quasi-invariant reference measure $\nu \sim \boldsymbol{\nu}$ on Γ.

THEOREM: *Let $\widetilde{P}$: $L^{\infty}(\widetilde{X}, \widetilde{m})\hookleftarrow$ be a covering Markov operator satisfying conditions* (P′) *and* (Stat), *and such that its quotient operator satisfies conditions* (Irr) *and* (Rec). *Let $\xi \preccurlyeq \zeta$ be two G-invariant measurable partitions of the Poisson boundary Γ of the operator $\widetilde{P}$ such that ζ is a refinement of ξ (here G is the deck group of the operator $\widetilde{P}$). Then the Radon–Nikodym cocycles Δ^{ξ} and Δ^{ζ} of the measures ν^{ξ} and ν^{ζ} are cohomologous over the space Γ^{ζ} if and only the partitions ξ and ζ coincide.*

The rest of §3.2 is devoted to a proof of this Theorem.

3.2.2 For the sake of simplicity we shall consider first the case when ζ is the point partition, so that the quotient space Γ^{ζ} coincides with the space Γ itself.

Suppose that the cocycles Δ and Δ^{ξ} are cohomologous. It means that there exists a positive measurable function φ on Γ such that

$$\Delta^{\xi}(g,\gamma) = \frac{\varphi(g\gamma)}{\varphi(\gamma)}\Delta(g,\xi) \qquad \forall g \in G\,,$$

or, in other words, that almost surely

$$\frac{dg(\varphi\nu)}{d(\varphi\nu)}(\gamma) = \frac{dg\nu^{\xi}}{d\nu^{\xi}}(\gamma^{\xi}) \qquad \forall g \in G\,.$$

Since the operator $\widetilde{P}$ has a Poisson kernel, the harmonic measures ν_x are absolutely continuous with respect to ν for $\widetilde{m}$-a.e. $x \in \widetilde{X}$. Hence, the quotient measures ν_x^ξ are also absolutely continuous with respect to the measure ν^ξ for almost all $x \in \widetilde{X}$.

3.2.3 Let λ_x, $x \in \widetilde{X}$ be the family of probability measures on Γ determined by their densities as

$$\frac{d\lambda_x}{d\nu}(\gamma) = \varphi(\gamma)\,\frac{d\nu_x^\xi}{d\nu^\xi}(\gamma^\xi)\,,$$

or, in other words,

$$\frac{d\lambda_x}{d(\varphi\nu)}(\gamma) = \frac{d\nu_x^\xi}{d\nu^\xi}(\gamma^\xi)\,.$$

Note that as it follows from (2), $\nu_x \prec \lambda_x$ for $\widetilde{m}$-a.e. $x \in \widetilde{X}$.

The family λ_x is G-invariant in the sense that almost surely $\lambda_{gx} = g\lambda_x$. Indeed,

$$\frac{d\lambda_{gx}}{d(\varphi\nu)}(\gamma) = \frac{d\nu_{gx}^\xi}{d\nu^\xi}(\gamma^\xi)\,,$$

and

$$\frac{dg\lambda_x}{dg(\varphi\nu)}(\gamma) = \frac{dg\nu_x^\xi}{dg\nu^\xi}(\gamma^\xi)\,,$$

whence (because $\nu_{gx} = g\nu_x$)

$$\frac{d\lambda_{gx}}{dg\lambda_x}(\gamma) = \frac{dg\nu^\xi}{d\nu^\xi}(\gamma^\xi)\Big/\frac{dg(\varphi\nu)}{d(\varphi\nu)}(\gamma) = 1$$

by definition of the function φ.

3.2.4 Let $\widetilde{\lambda}$ be a measure on the space $\widetilde{X} \times \Gamma$ defined as

$$d\widetilde{\lambda}(x,\gamma) = d\widetilde{m}(x)\,d\lambda_x(\gamma)\,.$$

The densities

$$\frac{d\lambda_x}{d\nu}(\gamma) = \varphi(\gamma)\,\frac{d\nu_x^\xi}{d\nu^\xi}(\gamma^\xi) = \varphi(\gamma)\int\frac{d\nu_x}{d\nu}(\gamma)\,d\nu^{\gamma^\xi}(\gamma)$$

are $\widetilde{P}$-harmonic as functions of x for a.e. $\gamma \in \Gamma$, so that by Proposition 1.4.4 the measure $\widetilde{\lambda}$ is a stationary measure of the extended operator $\check{\widetilde{P}}^\infty$. Being G-invariant, the measure $\widetilde{\lambda}$ determines a stationary measure λ of the quotient operator $\check{P}^\infty$. Since almost surely $\nu_x \prec \lambda_x$, the measure λ dominates the invariant measure m^∞ of the operator $\check{P}^\infty$. On the other hand, the operator $\check{P}^\infty$ is irreducible and recurrent with respect to the invariant measure m^∞ (Theorem 2.4.5), so that the Radon–Nikodym derivative $d\mu/d\lambda$ must be equal to a constant $K > 0$ almost everywhere with respect to the measure μ, which means that

$$\frac{d\nu_x}{d\lambda_x}(\gamma) = K$$

for $\widetilde{m}$-a.e. $x \in \widetilde{X}$ and ν_x-a.e. $\gamma \in \Gamma$. By definition of the measures λ_x the latter identity implies that

$$\frac{d\nu_x}{d\nu}(\gamma) = K\varphi(\gamma)\frac{d\nu_x^\xi}{d\nu^\xi}(\gamma^\xi) \tag{4}$$

for ν_x-a.e $\gamma \in \Gamma$. Note that so far we have used only the condition (P), and not its strengthening (P′).

Now, condition (P′) means that the harmonic measure ν_x is equivalent to ν for $\widetilde{m}$-almost all $x \in \widetilde{X}$. Thus, relation (4) holds for almost all $x \in \widetilde{X}$ and ν-almost all $\gamma \in \Gamma$. It means that the (minimal) harmonic functions corresponding to any two points γ, γ' with the same image γ^ξ are proportional, so that $\gamma = \gamma'$. Hence, the partition ξ is the point partition, and the space Γ^ξ coincides with Γ.

3.2.5 The general case can be treated along the same lines with replacing the extended operator $\widetilde{P}^\infty$ on the space $\widetilde{X} \times \Gamma$ with the analogous operator on the space $\widetilde{X} \times \Gamma^\xi$. It leads to the identity

$$\frac{d\nu_x^\zeta}{d\nu^\zeta}(\gamma) = K\varphi(\gamma)\frac{d\nu_x^\xi}{d\nu^\xi}(\gamma^\xi)$$

analogous to (4). Thus, if $\gamma_1^\xi = \gamma_2^\xi$, then the $\widetilde{P}$-harmonic functions $d\nu_x^\zeta/d\nu^\zeta(\gamma_1^\zeta)$ and $d\nu_x^\zeta/d\nu^\zeta(\gamma_2^\zeta)$ are proportional, which is only possible if $\gamma_1^\zeta = \gamma_2^\zeta$ (recall that points from Γ^ζ are in one-to-one correspondence with functions $d\nu_x^\zeta/d\nu^\zeta(\gamma^\zeta)$, because the minimal functions $d\nu_x/d\nu(\gamma)$ span a simplex).

3.3 Actions on the Poisson boundary.

3.3.1 THEOREM: *Let $\widetilde{P}$: $L^\infty(\widetilde{X}, \widetilde{m})\hookleftarrow$ be a covering Markov operator satisfying conditions* (P′) *and* (Stat), *and such that its quotient operator satisfies conditions* (Irr) *and* (Rec). *If $\xi \prec \zeta$ are two distinct G-invariant measurable partitions of the Poisson boundary $\Gamma(\widetilde{P})$, then almost all conditional measures corresponding to the projection $\Gamma^\zeta \to \Gamma^\xi$ are purely non-atomic.*

Proof: Once again for the sake of notational simplicity we shall consider only the case when ζ is the point partition if the space Γ. Let

$$A = \{\gamma \in \Gamma \colon \nu^{\gamma^\xi}(\gamma) > 0\}$$

be the subset of Γ consisting of all points γ which are atoms of the corresponding conditional measures. As it follows from (3), the set A is G-invariant. On the other hand, the G-action on Γ is ergodic by Theorem 2.1.4, so that the set A is either empty, or coincides with the whole space Γ (mod 0). In other words, almost all conditional measures ν^{γ^ξ} are either purely non-atomic or purely atomic simultaneously. Suppose that they are purely atomic. Then the function

$$\varphi(\gamma) = \nu^{\gamma^\xi}(\gamma)$$

is almost surely non-zero, and, as it follows from (3), it states an equivalence between the Radon–Nikodym cocycles Δ and Δ^ξ, which by Theorem 3.2.1 is only possible if ξ is the point partition. ∎

COROLLARY 1: *The Poisson boundary Γ and any its G-equivariant measurable quotient are either trivial of purely non-atomic.*

COROLLARY 2: *Let ζ be a G-invariant measurable partition of the Poisson boundary Γ. Then for any non-trivial G-invariant partition of the quotient Γ^ζ its elements are almost surely uncountable.*

COROLLARY 3: *The action of any finite normal subgroup $H \subset G$ on the Poisson boundary Γ is trivial.*

Proof: Let ξ_H be the partition of the Poisson boundary into ergodic components of the action of H. Since H is finite, the elements of this partition are H-orbits in Γ. These orbits being finite, ξ_H must be the point partition by Theorem 3.3.1, so that $H\gamma = \gamma$ for a.e. $\gamma \in \Gamma$. ∎

COROLLARY 4: *If the group G is generated by its finite normal subgroups, then the Poisson boundary Γ is trivial.*

Remark: Normality condition is essential in Corollaries 3 and 4 as it can be seen from examples of non-trivial Poisson boundaries for locally finite groups [K2] and for free products of finite groups [K3].

3.3.2 A group G is called **hyperfinite** if every homomorphic image $G' \neq \langle 1 \rangle$ of G has a normal finite subgroup $H \neq \langle 1 \rangle$. The class of hyperfinite groups is contained in the class of **locally finite** groups [KW], [Rob].

THEOREM: *Let* $\widetilde{P}$: $L^\infty(\widetilde{X}, \widetilde{m})\hookleftarrow$ *be a covering Markov operator satisfying conditions* (P′) *and* (Stat), *and such that its quotient operator satisfies conditions* (Irr) *and* (Rec). *If* H *is a hyperfinite normal subgroup of the deck transformations group* G, *then its action on the Poisson boundary* Γ *of the operator* $\widetilde{P}$ *is trivial. In particular, if* G *is hyperfinite, then* Γ *is trivial.*

Proof: If H is a finite normal subgroup of G, then by Corollary 3 of Theorem 3.3.1 and by Theorem 2.1.5 the Poisson boundary of the operator $\widetilde{P}$ coincides with the Poisson boundary of the operator $\widetilde{P}^H$ with the same quotient operator and with the deck group G/H. Now one can take a finite normal subgroup of the group G/H, etc. Thus, transfinite induction gives the desired statement. ∎

COROLLARY: *The Poisson boundary of the operator* $\widetilde{P}$ *coincides with the Poisson boundary of the operator* $\widetilde{P}^H$.

3.3.3 Recall that a measure type preserving action of a group H on a measure space (Ω, λ) is called **conservative** if it has no non-trivial **wandering** sets, i.e., such measurable sets $A \subset \Omega$ that all their translations gA, $g \in H$ are mutually disjoint (mod 0). If A is a non-trivial wandering set, then the set

$$\Omega_A = \bigcup_{g \in H} gA$$

is obviously H-invariant, and the ergodic components of the H-action on Ω_A are H-orbits (in particular, they are countable). Thus, we have

THEOREM: *Let* $\widetilde{P}$: $L^\infty(\widetilde{X}, \widetilde{m})\hookleftarrow$ *be a covering Markov operator satisfying conditions* (P′) *and* (Stat), *and such that its quotient operator satisfies conditions* (Irr) *and* (Rec). *Then the action of any normal subgroup of the deck group* G *on the Poisson boundary of the operator* $\widetilde{P}$ *is conservative.*

3.3.4. Remarks: 1. Quotients of the Poisson boundary of a random walk (G, μ) are known as μ-**boundaries** (or, **Furstenberg boundaries**) of the random walk, and the fact that they are either trivial or non-atomic is well-known [Fu2], [K3].

2. For regular covers of compact Riemannian manifolds Corollary 1 of Theorem 3.3.1 follows from the entropy theory [K4] (see Remark 2.2.3). In that particular case it was also later proved by Toledo [To] with an argument including a reference to the Tits' theorem on subgroups of linear groups.

3. An example of a G-invariant partition of the Poisson boundary which does not correspond to any normal subgroup of the deck group G is given by the group $G = \mathrm{SL}(n, \mathbb{Z})$ (or, more generally, any lattice in a semi-simple real Lie group of rank ≥ 2). Let $\widetilde{X} = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n, \mathbb{R})$ be the corresponding symmetric space, and $\widetilde{m}$ — the Riemannian volume on $\widetilde{X}$. Then the space $(\widetilde{X}, \widetilde{m})$ is a covering space, and the Brownian motion on $\widetilde{X}$ determines a covering Markov operator on $\widetilde{X}$. Its Poisson boundary coincides with the space of full flags in $\mathbb{R}^n$, the harmonic measure type being the smooth measure type [Ful], [Ka]. Any space of non-complete flags is a G-equivariant quotient of the Poisson boundary which does not correspond to any normal subgroup of G (they are all either finite or have a finite index in G). In fact, in this situation there are no G-equivariant quotients of the Poisson boundary others than the flag spaces [Ma].

4. The condition that H is a **normal** subgroup is essential in Theorem 3.3.3. For example, let $G = \mathbb{Z} * \mathbb{Z}$ be the free group with 2 generators a, b. Then for a large class of probability measures μ on G (including all finitely supported measures) the Poisson boundary Γ of the random walk (G, μ) can be identified with the set of infinite irreducible words in the alphabet $\{a, b, a^{-1}, b^{-1}\}$ with the corresponding harmonic measure [K3], [K9]. Thus, the action of the free factor $\mathbb{Z} \cong \{a^n\}$ on Γ is free and completely dissipative, the corresponding ergodic components being infinite irreducible words which begin with either b or b^{-1}.

4. Applications and examples

This Section is devoted to application of general methods developed above to more concrete situations. In §4.1 we use the fact that the Poisson extension of a corecurrent covering operator $\widetilde{P}$ is also corecurrent (hence, it has a unique stationary measure) to show that the center of the deck group acts trivially on the Poisson boundary of the operator $\widetilde{P}$ (Theorem 4.1.1). By transfinite induction it implies triviality of the Poisson boundary for corecurrent operators with hypercentral (in particular, nilpotent) deck groups (Theorem 4.1.4). Another application (§4.2) is to conformal densities of divergence type groups of hyperbolic motions. If the critical exponent δ satisfies the inequality $\delta \geq d/2$, then the conformal density is the harmonic measure of a corecurrent diffusion process on $\mathbb{H}^{d+1}$, so that results of Section 3 imply the rigidity of the corresponding Radon–Nikodym cocycles (Theorem 4.2.4). In particular, the action of any normal subgroup is conservative with respect to the conformal density.

In §4.3 we give simple examples of cotransient covering Markov operators with purely atomic Poisson boundary.

4.1 NILPOTENT COVERS.

4.1.1 Recall that the **center** $Z(G)$ of a group G is the set of all such elements $c \in G$ which commute with any element $g \in G$ (i.e., $cg = gc\,\forall\, g \in G$). The center is obviously a normal subgroup of G.

THEOREM: *Let $\widetilde{P}$: $L^\infty(\widetilde{X}, \widetilde{m})\hookleftarrow$ be a covering Markov operator satisfying conditions* (P) *and* (Stat), *and such that its quotient operator satisfies conditions* (Irr) *and* (Rec). *Then for any element c from the center $Z = Z(G)$ of the deck group G the following two conditions are equivalent:*

(1) *The action of c on the Poisson boundary Γ of the operator $\widetilde{P}$ is trivial, i.e., $c\gamma = \gamma$ for $\boldsymbol{\nu}$-a.e. $\gamma \in \Gamma$;*
(2) *The set $\{x \in \widetilde{X}: \nu_x \not\perp \nu_{cx}\}$ has non-zero measure $\widetilde{m}$.*

Proof: Clearly, if the harmonic measures ν_x and $\nu_{cx} = c\nu_x$ are singular, then $c\gamma \neq \gamma$ for ν_x-a.e. $\gamma \in \Gamma$, so that we have to prove only the implication (2) $\Rightarrow$ (1).

Let $\widetilde{\mu}_c$ be a measure on the space $\widetilde{X} \times \Gamma$ defined as

$$d\widetilde{\mu}_c(x,\gamma) = d\widetilde{m}(x)\, d\nu_{cx}(\gamma) = d\widetilde{m}(x)\, dc\nu_x(\gamma) = d\widetilde{m}(x)\, d\nu_x(c^{-1}\gamma)\,.$$

In particular, if $c = e$, then the measure $\widetilde{\mu}_e$ coincides with the invariant measure $\widetilde{m}^\infty$ of the extended Markov operator $\widetilde{P}^\infty$ on the space $\widetilde{X} \times \Gamma$ (see Definition 1.4.1). We shall show that for any $c \in Z$ the measure $\widetilde{\mu}_c$ is a G-invariant stationary measure of the operator $\check{\widetilde{P}}^\infty$.

Let ν be a reference probability measure on the Poisson boundary Γ of the operator $\widetilde{P}$. Then for a.e. $\gamma \in \Gamma$ the density

$$\frac{d\nu_{cx}}{d\nu}(\gamma) = \frac{dc\nu_x}{dc\nu}(\gamma)\,\frac{dc\nu}{d\nu}(\gamma) = \frac{d\nu_x}{d\nu}(c^{-1}\gamma)\,\frac{dc\nu}{d\nu}(\gamma)$$

is $\widetilde{P}$-harmonic, and the measure $\widetilde{\mu}_c$ is a stationary measure of the operator $\check{\widetilde{P}}^\infty$ by Proposition 1.4.4.

On the other hand, the measure $\widetilde{m}$ being G-invariant, for any $g \in G$

$$\begin{aligned} dg\widetilde{\mu}_c(x,\gamma) = d\widetilde{\mu}_c(g^{-1}x, g^{-1}\gamma) &= d\widetilde{m}(g^{-1}x)d\nu_{cg^{-1}x}(g^{-1}\gamma) \\ &= d\widetilde{m}(x)d\nu_{g^{-1}cx}(g^{-1}\gamma) \\ &= d\widetilde{m}(x)d\nu_{cx}(\gamma) = d\widetilde{\mu}_c(x,\gamma)\,. \end{aligned}$$

Thus, after factorization with respect to the G-action, the measure $\widetilde{\mu}_c$ determines a stationary measure μ_c of the operator $\check{P}^\infty$. Condition (2) means that the measures $\widetilde{m}^\infty$ and $\widetilde{\mu}_c$ are non-singular, hence the measures m^∞ and μ_c are also non-singular. Since m^∞ is a stationary measure of the irreducible recurrent operator $\check{P}^\infty$, this can only happen when $m^\infty \prec \mu_c$ and the Radon–Nikodym derivative $dm^\infty/d\mu_c$ equals a.e. to a constant $K > 0$. By definition of the measure μ_c it means that $\nu_x \prec \nu_{cx}$ for $\widetilde{m}$-a.e. $x \in \widetilde{X}$, and $d\nu_x/d\nu_{cx}(\gamma) = K$ for ν_x-a.e. $\gamma \in \Gamma$.

The same argument applied to the measure $\mu_{c^{-1}}$ shows that almost surely $\nu_x \prec \nu_{c^{-1}x}$, in other words, $\nu_{cx} \prec \nu_x$. Since ν_x are probability measures, the constant K must be 1, so that $\nu_x = \nu_{cx}$ for $\widetilde{m}$-a.e. $x \in \widetilde{X}$. Thus, the minimal harmonic functions $d\nu_x/d\nu(\gamma)$ and

$$\frac{d\nu_x}{d\nu}(c^{-1}\gamma) = \frac{d\nu}{dc\nu}(\gamma)\,\frac{d\nu_{cx}}{d\nu}(\gamma)$$

are proportional, so that they correspond to the same point $\gamma = c^{-1}\gamma$ of the Poisson boundary. ∎

4.1.2 We shall say that a subgorup H of the deck transformations group G of a covering Markov operator $\widetilde{P}\colon L^\infty(\widetilde{X},\widetilde{m})\hookleftarrow$ is **non-singular** if for $\widetilde{m}$-a.e. point $x \in \widetilde{X}$ the harmonic measures ν_x and ν_{gx} are non-singular for all $g \in H$.

Remark: If the operator $\widetilde{P}$ has transition densities, then non-singularity of harmonic measures ν_x and ν_{gx} is equivalent to non-singularity of the measure types $[\delta_x\colon \widetilde{P}]$ and $[\delta_{gx}\colon \widetilde{P}]$. In the general case, if the absolutely continuous parts of measure types $[\delta_x\colon \widetilde{P}]$ and $[\delta_{gx}\colon \widetilde{P}]$ are non-singular, then the harmonic measures ν_x and ν_{gx} are obviously also non-singular. On the other hand, singularity of measure types $[\delta_x\colon \widetilde{P}]$ and $[\delta_{gx}\colon \widetilde{P}]$ does not necessarily implies singularity of the harmonic measures ν_x and ν_{gx} (even being singular, these measure types are not necessarily separated by harmonic functions). For example, let $(\widetilde{X},\widetilde{m})$ be the real line with the Lebesgue measure on it, and the group $G = \mathbb{Z}$ acts on $\widetilde{X}$ by translations $x \mapsto x + n$. For an irrational number α the covering operator $\widetilde{P}f(x) = [f(x+\alpha) + f(x-\alpha)]/2$ has trivial Poisson boundary, whereas the measure types $[\delta_x\colon \widetilde{P}]$ and $[\delta_{x+1}\colon \widetilde{P}]$ are singular for all $x \in \mathbb{R}$.

Theorem 4.1.1 implies

THEOREM: *Let $\widetilde{P}\colon L^\infty(\widetilde{X},\widetilde{m})\hookleftarrow$ be a covering Markov operator satisfying conditions (P) and (Stat), and such that its quotient operator satisfies conditions*

(Irr) and (Rec). If the center Z of the group G is non-singular with respect to the operator $\widetilde{P}$, then the action of Z on the Poisson boundary Γ of the operator $\widetilde{P}$ is trivial, and the Poisson boundary of the operator $\widetilde{P}$ coincides with the Poisson boundary of the operator $\widetilde{P}^Z$. In particular, if the group G is abelian and non-singular, then the Poisson boundary Γ is trivial.

4.1.3 Let

$$G = G_0 \to G_1 \to \cdots$$

be the sequence of quotients of the group G defined inductively as

$$G_{n+1} = G_n / Z(G_n) ,$$

with $Z(G_n)$ being the center of the group G_n. In other words,

$$G_n = G/Z_n ,$$

where

$$\{e\} = Z_0 \subset Z_1 \subset \cdots$$

is the **upper central series** of the group G. Transfinite iteration of this construction gives the **transfinite upper series** $\{Z_\alpha\}$ of the group G. The terminal member H of the transfinite upper series is called the **hypercenter** of the group G. The group G is called **nilpotent** if there exists a finite number n such that $Z_n = G$, it is called ω-**nilpotent** if $Z_\omega = \bigcup Z_n = G$ (here ω is the first infinite ordinal), and it is called hypercentral if it coincides with its hypercenter H. Equivalently, a group G is hypercentral if every homomorphic image $G' \neq \langle 1 \rangle$ of G has a non-trivial center. Every hypercentral group is locally nilpotent, so that within the class of finitely generated groups hypercentrality is equivalent to nilpotency [KW], [Rob].

4.1.4 Transfinite induction applied to the transfinite upper series of the group G by Theorem 4.1.2 gives

THEOREM: *Let $\widetilde{P}$: $L^\infty(\widetilde{X}, \widetilde{m})\hookleftarrow$ be a covering Markov operator satisfying conditions* (P) *and* (Stat), *and such that its quotient operator satisfies conditions* (Irr) *and* (Rec). *If the hypercenter H of the group G is non-singular with respect to the operator $\widetilde{P}$, then the action of H on the Poisson boundary Γ of the operator $\widetilde{P}$ is trivial, and the Poisson boundaries of the operators $\widetilde{P}$ and $\widetilde{P}^H$ coincide. In particular, if G is hypercentral and non-singular, then the Poisson boundary Γ is trivial.*

4.1.5 One can go a little bit further by combining Theorem 4.1.2 with Theorem 3.3.2. We shall say that a group G is **hyper-finite-or-central** if every homomorphic image $G' \neq \langle 1 \rangle$ of G either has a non-trivial center or has a normal finite subgroup $H \neq \langle 1 \rangle$. A finitely generated hyper-finite-or-central group is a finite extension of a nilpotent group (cf. [Rob]).

THEOREM: *Let $\widetilde{P}: L^\infty(\widetilde{X}, \widetilde{m})\hookleftarrow$ be a covering Markov operator satisfying conditions* (P′) *and* (Stat), *and such that its quotient operator satisfies conditions* (Irr) *and* (Rec). *If H is a hyper-finite-or-central normal subgroup of G, then its action on the Poisson boundary Γ of the operator $\widetilde{P}$ is trivial, and the Poisson boundaries of the operators $\widetilde{P}$ and $\widetilde{P}^H$ coincide. In particular, if G is hyper-finite-or-central, then Γ is trivial.*

Remark: For regular covers of recurrent Riemannian manifolds Theorem 4.1.4 was proved in [LS] by using the Harnack inequality (in fact, in [LS] it is stated for ω-nilpotent covers only). Our proof uses a much milder condition (equivalence of harmonic measures on the Poisson boundary), which can be considered as a (very weak) form of the Harnack inequality at infinity. See §5.2 for a discussion of Theorem 4.1.4. in the case of random walks on groups. There is also a completely different approach to proving the Liouville theorem for nilpotent covers due to Lin [Li].

4.2 Conformal densities of divergence type groups.

4.2.1 The results obtained above for corecurrent operators can be also applied to covering operators with so-called **λ-recurrent** quotients. Let $P: L^\infty(X, m)\hookleftarrow$ be a Markov operator. For a number $t > 0$ let

$$G_t = (t - P)^{-1} = \sum_{n=0}^{\infty} \frac{P^n}{t^{n+1}}$$

be the corresponding **Green operator**. The number

$$\lambda = \lambda(P) = \sup\{t > 0: G_t \mathbf{1}_A \equiv \infty \;\forall\, A \subset X: m(A) > 0\}$$

is called the **convergence norm** of the operator P. The number $\lambda(P)$ can be also characterized as the infimum of all numbers $r > 0$ such that there exists a non-zero non-negative **r-superharmonic** function, i.e., such function f that $Pf \leq rf$.

If

$$G_\lambda \mathbf{1}_A \equiv \infty \qquad \forall\, A \subset X\colon m(A) > 0\,,$$

then the operator P is called λ-recurrent. For an irreducible λ-recurrent operator P there exists a unique (up to a multiplier) non-negative λ-harmonic function φ, and the Doob transform of the operator P corresponding to the function φ

$$P_\varphi = \frac{1}{\lambda}\mathbf{M}_\varphi^{-1} P \mathbf{M}_\varphi$$

is recurrent (here $\mathbf{M}_\varphi$ is the operator of multiplication by φ) [Tw].

4.2.2 Let now $\widetilde{P}\colon L^\infty(\widetilde{X},\widetilde{m})\hookleftarrow$ be a covering Markov operator such that its quotient operator $P\colon L^\infty(X,m)\hookleftarrow$ is λ-recurrent and irreducible. Denote by φ the (unique) λ-harmonic function of the operator P, and by $\widetilde{\varphi}$ its lift to $\widetilde{X}$. Then the operator P_φ is the quotient operator of a corecurrent covering operator $\widetilde{P}_{\widetilde{\varphi}}$. The Poisson boundary $\Gamma(\widetilde{P}_{\widetilde{\varphi}})$ of the operator $\widetilde{P}_{\widetilde{\varphi}}$ can be identified with the space $\mathcal{M}(\widetilde{P},\lambda)$ of minimal λ-harmonic functions of the operator $\widetilde{P}$. The harmonic measure ν_θ corresponding to an initial distribution θ is the representing measure of the function $\widetilde{\varphi}$ (normalized in such a way that $\langle\theta,\widetilde{\varphi}\rangle = 1$) in its decomposition into an integral of minimal λ-harmonic functions (with the same normalization condition). We shall denote by $\boldsymbol{\nu}_{\widetilde{\varphi}}$ the corresponding harmonic measure type, i.e., the common measure type of the measures ν_θ with $\theta \sim \widetilde{m}$.

Suppose that the operator $\widetilde{P}_{\widetilde{\varphi}}$ satisfies condition (P′). Then the results of previous Sections on the ergodic properties of the Poisson boundary of a corecurrent operator are applicable to the action of the deck group G on the space $(\mathcal{M}(\widetilde{P},\lambda),\boldsymbol{\nu}_{\widetilde{\varphi}})$.

4.2.3 Let us now recall some basic definitions concerning groups of isometries of hyperbolic spaces and conformal densities on their limit sets [N], [Pas], [Su].

Let G be a properly discontinuous group group of isometries of the hyperbolic space $\mathbb{H}^{d+1}$ (i.e., $\mathbb{H}^{d+1}$ is a simply connected Riemannian manifold of dimension $d+1$ with constant curvature -1). For convenience we shall fix a reference point $o \in \mathbb{H}^{d+1}$. The number

$$\delta = \delta(G) = \inf\{\alpha > 0\colon \sum_{g\in G} e^{-\alpha\,\mathrm{dist}(o,go)} < \infty\}$$

is called the **critical exponent** of the group G, and the group G is said to be of **divergence type** if

$$\sum_{g\in G} e^{-\delta\,\mathrm{dist}(o,go)} = \infty$$

(here $\operatorname{dist}(\cdot,\cdot)$ is the Riemannian distance on $\mathbb{H}^{d+1}$). The critical exponent δ always satisfies the inequality $\delta \le d$, and $\delta > 0$ for all non-elementary groups. The sphere $\mathbb{S}^d$ can be identified with the boundary of a natural **visibility** compactification of the space $\mathbb{H}^{d+1}$. In particular, it is endowed with an induced action of G. The closure in $\mathbb{H}^{d+1} \cup \mathbb{S}^n$ of the orbit Go is called the **limit set** of the group G. A probability measure σ on the limit set Λ is a **conformal density** of dimension α of the group G if it is quasi-invariant with respect to the action of G, and

$$\frac{dg\sigma}{d\sigma}(\gamma) = e^{-\alpha b_\gamma(go)} \qquad \forall g \in G,\ \gamma \in \Lambda\,,$$

where b_γ is the **Busemann function** of a point $\gamma \in \mathbb{S}^d$ with respect to the reference point o. There always exists a conformal density of dimension $\delta(G)$, and if the group G is of divergence type, then such conformal density is unique.

The Busemann function b_γ, $\gamma \in \mathbb{S}^d$ on $\mathbb{H}^{d+1}$ is defined as

$$b_\gamma(x) = \lim_{t\to\infty}\Big[\operatorname{dist}\big(x, \xi_\gamma(t)\big) - t\Big]\,,$$

where ξ_γ is the geodesic ray on $\mathbb{H}^{d+1}$ issued from o in the direction γ. Let m_x, $x \in \mathbb{H}^{d+1}$ be the (unique) probability measure on $\mathbb{S}^n$ invariant with respect to all isometries of $\mathbb{H}^{d+1}$ which fix the point x (i.e., m_x is the image of the Lebesgue measure on the sphere of the tangent space at x under the map assigning to a tangent vector the limit point of the corresponding geodesic). Then

$$\frac{dm_x}{dm_o}(\gamma) = e^{-db_\gamma(x)}\,.$$

The Poisson boundary of the Brownian motion on $\mathbb{H}^{d+1}$ can be identified with $\mathbb{S}^n$, the harmonic measure of a point $x \in \mathbb{H}^{d+1}$ being m_x. Thus, $e^{-db_\gamma(x)}$ is the Poisson kernel of the corresponding Markov operator with respect to the measure m_o.

4.2.4 THEOREM: *Let G be a properly discontinuous divergence type group of isometries of the hyperbolic space $\mathbb{H}^{d+1}$ with the critical exponent $\delta \ge d/2$, and let σ be the corresponding conformal density on the limit set Λ. Then the measure space (Λ, σ) has the following property: for any two distinct G-invariant partitions $\xi \prec \zeta$ their Radon–Nikodym cocycles are not cohomologous over the space $(\Lambda^\zeta, \sigma^\zeta)$. In particular, almost all conditional measures corresponding to the projection $\Lambda^\zeta \to \Lambda^\xi$ are purely non-atomic.*

Proof: First of all recall that the functions $e^{-\alpha b_\gamma}$ are eigenfunctions of the Laplace operator $\widetilde{\Delta}$ on $\mathbb{H}^{d+1}$ with the eigenvalue $-\alpha(d-\alpha)$, and they are minimal if $d/2 \le \alpha \le d$. The operator $\widetilde{\Delta}$ is the generating operator of the Brownian motion on $\mathbb{H}^{d+1}$ with the time 1 Markov transition operator $\widetilde{P} = e^{\widetilde{\Delta}}$.

Thus, the function

$$\widetilde{\varphi}(x) = \int e^{-\delta b_\gamma(x)}\, d\sigma(\gamma)$$

is a $e^{-\lambda}$-harmonic function of the operator $\widetilde{P}$, where $\lambda = \lambda(\delta) = \delta(d-\delta)$. The function $\widetilde{\varphi}$ is G-invariant, because

$$b_\gamma(g^{-1}x) = b_\gamma(g^{-1}x) - b_\gamma(o) = b_{g\gamma}(x) - b_{g\gamma}(go) ,$$

so that

$$\begin{aligned}\widetilde{\varphi}(g^{-1}x) &= \int e^{-\delta b_\gamma(g^{-1}x)}\, d\sigma(\gamma) \\ &= \int e^{-\delta[b_{g\gamma}(x) - b_{g\gamma}(go)]}\, d\sigma(\gamma) \\ &= \int e^{-\delta[b_{g\gamma}(x) - b_{g\gamma}(go)]}\, dg\sigma(g\gamma) \\ &= \int e^{-\delta b_{g\gamma}(x)}\, d\sigma(g\gamma) = \widetilde{\varphi}(x) .\end{aligned}$$

Hence, $\widetilde{\varphi}$ is the lift of a $e^{-\lambda}$-harmonic function φ of the quotient operator $P = e^{\Delta}$ (here Δ is the Laplacian on the quotient manifold $\mathbb{H}^{d+1}/G$).

On the other hand, the Green function $\widetilde{G}_\lambda(x,y)$ of the operator $-\widetilde{\Delta}$ has the asymptotic

$$\widetilde{G}_\lambda(x,y) \sim e^{-\left(\frac{d}{2} + \sqrt{\frac{d^2}{4} - \lambda}\right)\operatorname{dist}(x,y)}$$

when $\operatorname{dist}(x,y) \to \infty$. In particular, for $\alpha \ge \frac{d}{2}$

$$\widetilde{G}_{\alpha(d-\alpha)}(x,y) \sim e^{-\alpha \operatorname{dist}(x,y)} .$$

Since the Green function of the operator Δ is obtained from the Green function of the operator $\widetilde{\Delta}$ by the formula

$$G_\lambda(x,y) = \sum_{g\in G} G_\lambda(\widetilde{x}, g\widetilde{y}) ,$$

where x, y are projections of points $\widetilde{x}, \widetilde{y}$ from $\mathbb{H}^{d+1}$ onto the quotient manifold $\mathbb{H}^{d+1}/G$, it means that the convergence norm of the Markov operator $P = e^{-\Delta}$ is $e^{-\delta(d-\delta)}$, and it is $e^{-\delta(d-\delta)}$-recurrent.

Now we can apply Theorem 3.2.1 to the covering corecurrent operator

$$\widetilde{P}_{\widetilde{\varphi}} = e^{\delta(d-\delta)}\mathbf{M}_{\widetilde{\varphi}}^{-1}\widetilde{P}\mathbf{M}_{\widetilde{\varphi}} . \qquad \blacksquare$$

COROLLARY 1: *Any non-trivial G-equivariant measurable quotient of the space (Λ, σ) is purely non-atomic.*

COROLLARY 2: *The action of any normal subgroup of the group G on the space (Λ, σ) is conservative.*

4.2.5 Remarks: 1. The statement of Corollary 1 is well known for the space (Λ, σ) itself (conformal density of divergence type groups does not have atoms).

2. For the case when $\delta = d$ (i.e., when the manifold $\mathbb{H}^{d+1}/G$ is recurrent) the statement of Corollary 2 was proved by Velling and Matsuzaki [VM] (see also [Ta], [V]).

3. The assumption $\delta \geq d/2$ is essential in our proof of Theorem 4.2.4 (as it was already pointed out by Sullivan [Su], the probability methods do not readily apply to the case when $\delta < d/2$). It would be interesting to find out whether Theorem 4.2.4 is true in this case as well.

4. The operator $\widetilde{P}_{\widetilde{\varphi}}$ is the time 1 transition operator of the diffusion process on $\mathbb{H}^{d+1}$ with the generating operator

$$\log \widetilde{P}_{\widetilde{\varphi}} = \mathbf{M}_{\widetilde{\varphi}}^{-1}\widetilde{\Delta}\mathbf{M}_{\widetilde{\varphi}} + \delta(d-\delta)I = \widetilde{\Delta} + 2\nabla \log \widetilde{\varphi} ,$$

where $\widetilde{\Delta}$ is the Laplace operator on $\mathbb{H}^{d+1}$, and I is the identity operator. Thus, Theorem 5.1.6 shows that for $\delta \geq d/2$ there exists a probability measure μ on G such that the conformal density σ coincides with the harmonic measure of the random walk (G, μ), so that Theorem 4.2.4 can be deduced from results of §5.2 on Poisson boundaries of random walks without making recourse to general results from Section 3.

4.3 COTRANSIENT OPERATORS.

4.3.1 We shall say that a covering Markov operator $\widetilde{P}$: $L^\infty(\widetilde{X}, \widetilde{m})\hookleftarrow$ is **cotransient** if its quotient operator P is transient. We shall give here simple examples which show that for cotransient covering operators the Poisson boundary of the covering chain can be purely atomic non-trivial for an arbitrary deck transformation group G. In particular, for a non-amenable G it means that the

Poisson boundary for covering chains can be "smaller" (atomic vs. purely non-atomic) in cotransient case than in the corecurrent case, although the quotient state space is in a sense "larger" in the cotransient case.

4.3.2 Definition: A covering Markov operator $\widetilde{P}$: $L^\infty(\widetilde{X},\widetilde{m})\hookleftarrow$ is **almost disjoint** if there exists a subset $X_d \subset \widetilde{X}$ such that

(a) All translations gX_d, $g \in G$ are mutually disjoint (mod 0);

(b) The set $\widetilde{X}_t = \widetilde{X} \setminus \bigcup gX_d$ is transient for the operator $\widetilde{P}$;

(c) If $f \in L^\infty(\widetilde{X},\widetilde{m})$ is a function supported by the set X_d, then $\widetilde{P}f \equiv 0$ on all sets gX_d, $g \in G \setminus \{e\}$.

In other words, the operator $\widetilde{P}$: $L^\infty(\widetilde{X},\widetilde{m})\hookleftarrow$ is almost disjoint if the corresponding Markov chain can get from g_1X_d to g_2X_d, $g_1 \neq g_2$ only by passing through the transient set $\widetilde{X}_t$. Clearly, any almost disjoint covering operator is cotransient.

4.3.3 THEOREM: *The action of the deck transformations group G on the Poisson boundary $\Gamma(\widetilde{P})$ of an almost disjoint covering Markov operator $\widetilde{P}$: $L^\infty(\widetilde{X},\widetilde{m})$ $\hookleftarrow$ is completely dissipative. In particular, if the Poisson boundary $\Gamma(P)$ of the quotient operator is trivial, then $\Gamma(\widetilde{P})$ coincides with the group G.*

Proof: By Theorem 2.1.4 it is sufficient to consider only the case when the Poisson boundary $\Gamma(P)$ of the quotient operator P is trivial, and to show that in this case the Poisson boundary of the covering operator can be identified with G. Take a reference measure $\theta \sim \widetilde{m}$ on $\widetilde{X}$. Since the operator $\widetilde{P}$ is almost disjoint, for a.e. path $\overline{x} = \{x_n\}$ there exists $g = g(\overline{x}) \in G$ such that $x_n \in gX_d$ for all sufficiently large n. Since the measure θ on $\widetilde{X}$ is G-quasi-invariant, the distribution of $g(\{x_n\})$ is also quasi-invariant, i.e., is equivalent to the counting measure m_G. The fact that G coincides with the *whole* Poisson boundary of the operator $\widetilde{P}$ follows from triviality of the Poisson boundary of the operator P. ∎

4.3.4 We shall give now a simple example of an almost disjoint covering Markov chain (similar to an example from [PW1]).

Example: Let P: $L^\infty(X,m)\hookleftarrow$ be a transient Markov operator on a countable state space X with transition probabilities $\{p(x,y)\colon x,y \in X\}$ (here m is the counting measure on X). Let $o \in X$ be a state such that $p(o,o) = \varepsilon > 0$. Take a discrete group G with a probability measure μ on it and consider a new Markov operator $\widetilde{P}$ with the state space $G \times X$ and transition probabilities $\widetilde{p}(\cdot,\cdot)$ defined as:

$$\begin{aligned}
\widetilde{p}\big((g,x),(g,y)\big) &= p(x,y)\,, & x \neq o\,,\\
\widetilde{p}\big((g,o),(g,y)\big) &= p(o,y)\,, & y \neq o\,,\\
\widetilde{p}\big((g,o),(gh,o)\big) &= \varepsilon\mu(h)\,, & h \in G\,.
\end{aligned}$$

One can visualize the state space $G \times X$ in the following way: for each $g \in G$ there is a copy $X_g = \{g\} \times X$ of the state space X "sticking out" of the point (g,o), so that one can pass from one copy to another only by passing through the "stem" $(g,o) \in X_g$. The operator $\widetilde{P}$ is an almost disjoint covering operator of the operator P.

4.3.5 The next example shows that there exist covering cotransient operators such that their Poisson boundary has both countable and continious G-ergodic components. It is obtained by combining the product of the underlying chain and a random walk on the deck group (which gives continuous ergodic components) with the construction from Example 4.3.4.

Example: Let P be a transient Markov operator with a countable state space X. Suppose that there exists a state $o \in X$ such that $X \setminus \{o\}$ can be divided into two subsets X_-, X_+ with the property that

$$p(x,y) = 0 \qquad \forall\,(x,y) \in X_- \times X_+ \cup X_+ \times X_-\,.$$

In other words, one can get from X_- to X_+ or from X_+ to X_- only by passing through o. Suppose, further, that the probabilities of escaping to infinity staying in X_- (resp., in X_+) are both positive. Denote by Γ_- and Γ_+ the corresponding subsets of the Poisson boundary of the operator P.

Take a disrete group G with a probability measure μ and define a new operator $\widetilde{P}$ with the state space $G \times X$ and transition probabilities $\widetilde{p}(\cdot,\cdot)$:

$$\begin{aligned}
\widetilde{p}\big((g,x),(g,y)\big) &= p(x,y)\,, & x \in X_-\,,\\
\widetilde{p}\big((g,o),(g,y)\big) &= p(o,y)\,, & y \in X_-\,,\\
\widetilde{p}\big((g,o),(gh,y)\big) &= \mu(h)\,p(o,y)\,, & y \in X_+ \cup \{o\}\,,\\
\widetilde{p}\big((g,x),(gh,y)\big) &= \mu(h)\,p(x,y)\,, & x \in X_+\,.
\end{aligned}$$

The constructed covering Markov operator is almost disjoint when restricted to $G \times (X_- \cup \{o\})$ and coincides with the product of the random walk (G,μ) and the chain on $X_+ \cup \{o\}$ when restricted to $G \times (X_+ \cup \{o\})$. For the chain on $G \times (X_- \cup \{o\})$ the Poisson boundary is $\widetilde{\Gamma}_- = G \times \Gamma_-$ (see Example 4.3.4), whereas the Poisson boundary of the latter product chain coincides with the product of the Poisson boundaries $\Gamma(G,\mu)$ and Γ_+ [Mo]. Hence, the Poisson boundary of the operator $\widetilde{P}$ is the disjoint union

$$\widetilde{\Gamma} = G \times \Gamma_- \bigcup \Gamma(G,\mu) \times \Gamma_+ .$$

Thus, among the G-ergodic components in $\widetilde{\Gamma}$ are both discrete (on $G \times \Gamma_-$) and continuous (on $\Gamma(G,\mu) \times \Gamma_+$) components.

4.3.6 Remarks: 1. Example 4.3.4 shows that even *finite* covers of Liouville chains can have a non-trivial Poisson boundary. It also shows that if the deck transformations group G is non-amenable, then for "larger" (transient) quotient spaces the Poisson boundary can be "smaller" (countable vs. uncountable) than for "smaller" (recurrent) ones. In a certain sense this phenomenon is analogous to what happens with the spectrum of the Laplacian $\widetilde{\Delta}$ on a covering Riemannian manifold. If the quotient manifold M is compact, then $0 \in \operatorname{spec} \widetilde{\Delta}$ if and only if the deck transformations group G is amenable [Br]. On the other hand, for non-compact M the spectrum $\operatorname{spec} \widetilde{\Delta}$ can contain zero even if G is non-amenable, so that again non-amenable covers of larger manifolds can be "more amenable" than covers of smaller manifolds.

2. Examples analogous to Examples 4.3.4 and 4.3.5 can be also constructed within the class of simple random walks on graphs, and extended to covering manifolds using a correspondence between random walks on graphs and the Brownian motion on specially chosen Riemannian surfaces [Lt] (or such examples can be constructed directly for Riemannain manifolds). If all vertices of a graph X have the same degree, then the simple random walk on the product of X and a Cayley graph of a finitely generated group G coincides with the **Cartesian product** of simple random walks on X and G, i.e., its transition operator has the form $P = \alpha P_X \otimes I_G + (1-\alpha) I_X \otimes P_G$, where P_X and P_G are Markov operators of simple random walks on X and G, respectively, I_X and I_P are unit operators, and $0 < \alpha < 1$. The Poisson boundary of a Cartesian product is isomorphic to the product of Poisson boundaries of the factors [PW2].

3. It would be interesting to understand better which covering operators (or, covering manifolds, for the Brownian motion case) have purely atomic non-trivial Poisson boundary. Notice that the Poisson boundary can not have non-trivial atoms for the Brownian motion on negatively curved simply connected manifolds with pinched curvature [KL] (see also [K10]) and for simple random walks on trees [BP], [Lr]. On the other hand, the Riemannian surfaces arising from Example 4.3.4 lead to covers of hyperbolic space forms with an arbitrary deck group G and atomic non-trivial Poisson boundary. One can also construct an easy example of a universal covering manifold with atomic Poisson boundary (G. Mess). Excise from the Euclidean space $\mathbb{R}^3$ interiors of 2 non-intersecting closed balls, and connect their boundary spheres by a "handle" $\mathbb{S}^2 \times [0,1]$. Then the fundamental group of the resulting manifold is $\mathbb{Z}$, and the Brownian motion on its universal covering manifold is almost disjoint (since $\mathbb{R}^3$ is transient). It is unclear, whether a manifold with a purely atomic non-trivial Poisson boundary can be constructed within the class of universal covers of non-positively curved manifolds or even just within the class of Cartan–Hadamard manifolds (without any additional covering structure). See [Av], [Do], [KM], [LT] for examples of Riemannian manifolds with a non-trivial finite Poisson boundary.

5. Covering Markov operators and random walks on groups

In this Section we consider interrelations between general covering operators and the simplest possible covering operators which correspond to random walks on countable groups. In §5.1 we show that for two classes of corecurrent operators (operators on a discrete state space and operators corresponding to diffusion processes) their Poisson boundary coincides with the Poisson boundary of an appropriate random walk on the deck group, so that the covering operator is in a sense approximated by the random walk on the deck group. It would not be reasonable to expect to have such an approximation for cotransient operators (their sample paths leaving almost surely the orbit of any finite measure set), and examples of almost disjoint operators (§4.3) show that, indeed, there are cotransient operators such that their Poisson boundary can never serve as the Poisson boundary of a random walk on the deck group.

Considering random walks on groups instead of general corecurrent operators makes proofs of results from Sections 2 and 3 much simpler. For reader's convenience we give these proofs in §5.2. In view of §5.1, this generality is sufficient to deal with the Poisson boundaries of corecurrent diffusion processes.

5.1 APPROXIMATION OF COVERING OPERATORS BY RANDOM WALKS ON GROUPS.

5.1.1 We shall begin with corecurrent Markov operators on a discrete state space $\widetilde{X}$.

THEOREM: *Let $\widetilde{P}$: $L^\infty(\widetilde{X},\widetilde{m})\hookleftarrow$ be a corecurrent Markov operator on a countable covering space $\widetilde{X}$ such that any two points from $\widetilde{X}$ communicate. Then for any reference point $o \in \widetilde{X}$ there exists a probability measure μ on the deck group G with the following property. The space of bounded $\widetilde{P}$-harmonic functions $H^\infty(\widetilde{X},\widetilde{P})$ and the space $H^\infty(G,\mu)$ of bounded μ-harmonic functions on the group G are isometric: for any bounded $\widetilde{P}$-harmonic function its restriction onto the orbit $Go \cong G$ is μ-harmonic, and, conversely, any bounded μ-harmonic function on the orbit Go can be uniquely extended to a bounded $\widetilde{P}$-harmonic function.*

Proof: Corecurrence of the operator $\widetilde{P}$ in combination with its irreducibility means that a.e. path of the corresponding Markov chain visits the orbit Go which we shall identify with the group G by using the map $g \mapsto go$. Now, it is a general fact from the theory of Markov operators that the induced operator on a recurrent subset has the same space of bounded harmonic functions as the original one. For the reader's convenience we shall give a detailed argument which would also simplify understanding of a more complicated construction of Theorem 5.1.6.

Let μ_x, $x \in \widetilde{X}$ be the hitting distribution on the orbit $Go \cong G$ corresponding to a starting point $x \in \widetilde{X}$. Then the family of measures μ_x is G-invariant in the sense that

$$\mu_{gx} = g\mu_x \qquad \forall\, x \in \widetilde{X},\, g \in G\ ,$$

and the harmonic measures on the Poisson boundary of the operator $\widetilde{P}$ satisfy the relation

$$\nu_x = \sum_{g\in G} \mu_x(g)\nu_{go} = \sum_{g\in G} \mu_x(g) g\nu_o \qquad \forall\, x \in \widetilde{X}\ .$$

In particular, if $x = o$, then the measure $\nu = \nu_o$ satisfies the stationarity relation $\nu = \mu\nu$. Thus, for any bounded $\widetilde{P}$-harmonic function f with boundary values $\widehat{f}$ on the Poisson boundary

$$f(go) = \langle \widehat{f}, \nu_{go}\rangle = \langle \widehat{f}, g\nu\rangle = \langle \widehat{f}, g\mu\nu\rangle = \sum \mu(h)\langle \widehat{f}, gh\nu\rangle = \sum \mu(h) f(gho)\ ,$$

so that the restriction of any bounded $\widetilde{P}$-harmonic function onto the orbit $Go \cong G$ is a μ-harmonic function.

Conversely, the measures μ_x satisfy the relation

$$\mu_x = \sum_{y \in \widetilde{X} \setminus Go} \widetilde{p}(x,y)\mu_y + \sum_{g \in G} \widetilde{p}(y, go)\delta_g \,,$$

where $\widetilde{p}(\cdot,\cdot)$ are transition probabilities of the operator $\widetilde{P}$. Thus, if $\nu' = \nu'_e$ is the harmonic measure on the Poisson boundary Γ' of the random walk (G, μ) corresponding to the initial distribution δ_e, then the measures $\nu'_x = \mu_x \nu'$ on Γ' satisfy the identity

$$\nu'_x = \sum_{y \in \widetilde{X}} \widetilde{p}(x,y)\nu'_y \,.$$

Now, if $f \in L^\infty(G, \mu)$ is a bounded μ-harmonic function, and $\widehat{f}$ – the corresponding function on the Poisson boundary Γ', then its extension to $\widetilde{X}$ defined as

$$f(x) = \sum_g f(go)\mu_x(g) = \langle \widehat{f}, \nu'_x \rangle$$

is $\widetilde{P}$-harmonic. ∎

Corollary: *The Poisson boundaries of the operator $\widetilde{P}$ and of the random walk (G, μ) coincide.*

5.1.2 If the space $(\widetilde{X}, \widetilde{m})$ is non-atomic, the argument above is not applicable, because the probability to hit any countable set (in particular, the orbit Go) is zero. Nonetheless, if the operator $\widetilde{P}$ is the transition operator of a **continuous time** Markov process with **continuous paths** on a topological covering space $\widetilde{X}$, then one can still construct a random walk on the deck group G with the same Poisson boundary.

The construction below was first proposed by Furstenberg [Fu2] for the Brownian motion on symmetric spaces and later generalized by Lyons and Sullivan [LS]. As it was proved in [Fu2] and [LS], for any bounded $\widetilde{P}$ harmonic function its restriction onto the orbit Go is μ-harmonic for the constructed measure μ, so that the Poisson boundary $\Gamma(\widetilde{P})$ is a quotient of the Poisson boundary $\Gamma(G, \mu)$. In fact, one can show that these Poisson boundaries coincide [K6]. For the sake of completeness we shall reproduce here the argument from [K6] (in a slightly modified form). Note that yet another discretization procedure was proposed by

Ancona [An]. In his construction the operator $\widetilde{P}$ and the random walk (G,μ) have the same *positive* harmonic functions.

5.1.3 Let $\widetilde{X}$ be a topological space with a properly discontinuous action of a countable group G, and let $\widetilde{P}\colon L^\infty(\widetilde{X},\widetilde{m})\hookleftarrow$ be a Markov operator which is the time 1 transition operator of a G-invariant continuous time Markov process on $\widetilde{X}$ with (almost surely) continuous sample paths. For a closed set $A\subset\widetilde{X}$ and a point $x\notin A$ let $\varepsilon_x^A=\varepsilon(x,A)$ be the corresponding **harmonic measure** on the boundary ∂A, i.e., the distribution of points where trajectories starting from the point x for the first time meet the set A. We shall impose on the operator $\widetilde{P}$ the following condition

(Har) There exist two sets $E\subset V\subset\widetilde{X}$ such that

- (a) The set E is compact, $\widetilde{m}(E)>0$, and for any point $x\in\widetilde{X}$ a.e. trajectory starting from x visits the union $\bigcup_g gE$;
- (b) The set V is open;
- (c) The translations $g\overline{V}$, $g\in G$ of the closure $\overline{V}$ are mutually disjoint;
- (d) For all points $x\in E$ the harmonic measures $\varepsilon_x^{\complement V}$ on the complement $\complement V$ of the set V have mass 1, are mutually absolutely continuous, and there exists a constant $C>0$ such that

$$\varepsilon_x^{\complement V}\le C\varepsilon_y^{\complement V}\qquad\forall\, x,y\in E$$

(a **Harnack inequality**).

5.1.4 We shall now describe a discretization procedure for the operator $\widetilde{P}$ (or, more precisely, for the continuous time Markov process on $\widetilde{X}$ with sample paths $\xi=\{\xi(t)\}_{t\ge0}$). Let us fix a reference point $o\in E$. For a point $x\in\bigcup_g g\overline{V}$ let $g(x)\in G$ be the group element uniquely determined by the condition $x\in g(x)\overline{V}$, and let $V_x=g(x)V$.

Let $(R_n)_{n\ge1}$ and $(S_n)_{n\ge0}$ be the Markov stopping times of the process $\{\xi_t\}$ defined as

$$S_0(\xi)=\begin{cases}0 & ,\ \xi(0)\notin Go\,,\\ \min\{t>0\colon \xi(t)\in\widetilde{X}\setminus V_{\xi(0)}\} & ,\ \xi(0)\in Go\,,\end{cases}$$

and

$$R_n(\xi)=\min\{t>S_{n-1}(\xi)\colon \xi(t)\in\bigcup_g gE\}$$

$$S_n(\xi)=\min\{t>R_n(\xi)\colon \xi(t)\in\widetilde{X}\setminus V_{\xi(R_n)}\}\,.$$

Put also

$$g_n(\xi) = g(\xi(R_n)) \in G .$$

Let $\alpha = (\alpha_n)_{n\geq 1}$ be a sequence of i.i.d. random variables which are independent of the process $\{\xi(t)\}$ and have the Lebesgue measure on the unit interval $(0,1)$ as their common distribution. Put

$$N_0(\xi,\alpha) = 0 ,$$

and for $k \geq 1$ by induction

$$N_k(\xi,\alpha) = \min\left\{n > N_{k-1}\colon \alpha_n < \frac{1}{C}\,\frac{d\varepsilon(g_n o, \complement V_{g_n o})}{d\varepsilon(\xi(R_n), \complement V_{g_n o})}\,(\xi(S_n))\right\} .$$

Now for every point $x \in \widetilde{X}$ let

$$\mu_x(g) = {}_x\mathbf{P}\,[g_{N_1} = g]$$

be the probability measure on G which is the distributions of the first coordinate g_{N_1} of the random G-valued sequence $(g_{N_k})_{k\geq 1}$ for the sample paths ξ starting from x. Denote by $\mu = \mu_o$ the probability measure on G corresponding to the point o.

5.1.5 The measures μ_x can be also described in the following way. Start with the distribution δ_x. If $x \in Go$, *balayage* δ_y onto $\complement V_x$ (this corresponds to the stopping time S_0). Now apply to the resulting measure λ_0 the following recurrent procedure (we describe the inductive step for an arbitrary measure λ_n). *Balayage* λ_n first onto $\bigcup_g gE$ (it corresponds to the stopping time R_n). Let $\lambda_n^g, g \in G$ be the restrictions of λ_n onto the sets gE, and $\widetilde{\lambda}_n^g$ be the balayage of the measure λ_n^g onto the set $\complement V_{go}$ (it corresponds to the stopping time S_n). Now put

$$\lambda_{n+1} = \sum_{g\in G}\left(\widetilde{\lambda}_n^g - \frac{\|\widetilde{\lambda}_n^g\|}{C}\,\varepsilon(go, \complement V_{go})\right) ,$$

and

$$\tau_n = \sum_{g\in G} \frac{\|\widetilde{\lambda}_n^g\|}{C}\,\delta_g .$$

Finally, the sought for measure μ_x is presented as the sum

$$\mu_x = \sum_{n\geq 0} \tau_n .$$

5.1.6 THEOREM: *Under condition* (Har) *the space $H^\infty(\widetilde{X},\widetilde{P})$ of bounded $\widetilde{P}$-harmonic functions and the space $H^\infty(G,\mu)$ of bounded μ-harmonic functions on G are isometric: for any bounded $\widetilde{P}$-harmonic function its restriction onto the orbit $Gx \cong G$ is μ-harmonic, and, conversely, any bounded μ-harmonic function on the orbit Gx can be uniquely extended to a bounded $\widetilde{P}$-harmonic function. In particular, the Poisson boundaries of the operator $\widetilde{P}$ and of the random walk (G,μ) coincide.*

Proof: The general idea of the proof is basically the same as for Theorem 5.1.1. Clearly, the system of measures μ_x is G-invariant (because the whole construction is G-invariant). Since *balayage* does not change the harmonic measure on the Poisson boundary, the measures μ_x satisfy the identity

$$\nu_x = \sum \mu_x(g)\nu_{go} = \sum \mu_x(g)g\nu_o = \mu_x\nu_o \qquad \forall y \in \widetilde{X} .$$

In particular, if $\mu = \mu_o$, and $\nu = \nu_o$, then $\nu = \mu\nu$ (the measure ν is μ-**stationary**). Thus, the restriction of any bounded $\widetilde{P}$-harmonic function onto the orbit Go is μ-harmonic.

The proof of the converse statement, that any bounded μ-harmonic function can be extended to a $\widetilde{P}$-harmonic function is more complicated than in the discrete case. What we are going to prove is that all bounded μ-harmonic functions are restrictions of $\widetilde{P}$-harmonic functions, or, in probabilistic terms, that the Poisson boundary of the random walk (G,μ) is a quotient of the Poisson boundary of the operator $\widetilde{P}$. By the argument in the first part of the proof this would imply that these Poisson boundaries in fact coincide.

First of all note that for a given $g_{N_k} = g$ the distribution of $\xi(S_{N_k})$ is the measure $\varepsilon(go, \complement V_{go})$ which depends on the value of g only, so that g_{N_k} is the (Markov) random walk on the group G governed by the measure μ. Thus, the random walk (G,μ) is obtained from the Markov process $\{\xi(t)\}$ on $\widetilde{X}$ as a result of the following series of consecutive measure preserving mappings of Markov processes:

$$\begin{aligned}\{\xi_t\} &\xrightarrow{1} \{\xi(R_n),\xi(S_n)\} \xrightarrow{2} \{\xi(R_n),\xi(S_n),\alpha_n\} \\ &\xrightarrow{3} \{\xi(R_{N_k}),\xi(S_{N_k}),\alpha_{N_k}\} \xrightarrow{4} \{(g_{N_k}\}.\end{aligned}$$

On the first step we obtain a discrete time process with the state space $\widetilde{X}\times\widetilde{X}$, then we add the sequence $\{\alpha_n\}$, pass to the subsequence of times (N_k), and finally take the images $g_{N_k} = g(\xi(R_{N_k}))$. The processes obtained on the steps

(1) and (2) are evidently Markov, because the stopping times R_n and S_n are Markov and the sequence (α_n) is i.i.d. and independent of the process $\{\xi(t)\}$. The process obtained on step (3) can be considered as the process induced by the process $(\xi(R_n), \xi(S_n), \alpha_n)$ on the recurrent set

$$\left\{\alpha_n < \frac{1}{C} \frac{d\varepsilon(g_n o, \complement V_{g_n o})}{d\varepsilon(\xi(R_n), \complement V_{g_n o})} (\xi(S_n))\right\}.$$

Recall that the Poisson boundary of a Markov process $(Z_t)_{t\geq 0}$ (i.e., the space of ergodic components of the shift in its state space) is the factor-space of its path space with respect to the measurable envelope of the **stationary equivalence relation**: two trajectories (Z_t^1) and (Z_t^2) are equivalent iff there exist $T_1, T_2 > 0$ such that $Z^1_{T_1+t} = Z^2_{T_2+t}$ for all $t \geq 0$.

We have to show that each of the mappings (1)–(4) does not extend the Poisson boundary, so that the Poisson boundary of any subsequent process is a quotient of the Poisson boundary of the preceding one.

If two sample paths $\{\xi_t^1\}$ and $\{\xi_t^2\}$ are stationary equivalent, then the corresponding sequences $\{\xi(R_n^i), \xi(S_n^i)\}, i = 1,2$ also are. The mapping (2) does not change the space of bounded harmonic functions (α_n are independent, hence the transition probabilities of the obtained process do not depend on α_n). On step (3) we get an induced process on a recurrent set, hence the Poisson boundary does not change on this step either. Finally, on step (4) the new Markov process is obtained by factorizing the state space of the preceding one by the map

$$\left((\xi(R_{N_k}), \xi(S_{N_k}), \alpha_{N_k}\right) \mapsto g_{N_k} = g\left(\xi(R_{N_k})\right),$$

so that the Poisson boundary of the new process must be a quotient of the Poisson boundary of the old one. ∎

5.2 Random walks on groups.

5.2.1 The simplest possible covering Markov operator is one with the trivial quotient operator P on a trivial (i.e., consisting of a single point) space X. In this case the covering space $(\widetilde{X}, \widetilde{m})$ can be identified with the deck group G, where the measure $\widetilde{m}$ is the Haar (counting) measure m_G on G. The covering operator $\widetilde{P}$ is G-invariant, which means that it has the form $\widetilde{P} = P_\mu$ with

$$\widetilde{P}_\mu f(g) = \sum f(gx)\mu(x),$$

where μ is a probability measure on G. In other words, $\widetilde{P} = P_\mu$ is the Markov operator of the (right) **random walk** on the group G determined by the measure μ.

Obviously, the trivial Markov operator on the one-point space satisfies conditions (Irr) and (Rec), and the random walk operator P_μ satisfies condition (Stat) with respect to the Haar measure m_G. The reversed operator of P_μ is the operator $\check{P}_\mu = P_{\check{\mu}}$ corresponding to the reflected measure $\check{\mu}(g) = \mu(g^{-1})$. Since the state space G is countable, the operator P_μ has a Poisson kernel, i.e., it satisfies condition (P). Let $\Gamma = \Gamma(G,\mu)$ be the Poisson boundary of the operator P_μ, and $\nu = \nu_e$ — the harmonic measure on Γ corresponding to the initial distribution δ_e concentrated on the identity e of the group G. Then for an arbitrary initial distribution θ on G the corresponding harmonic measure on Γ has the form

$$\nu_\theta = \sum \theta(g)\nu_g = \sum \theta(g) g\nu = \theta\nu \ .$$

In particular, since $\nu = \nu_\mu$, the measure ν is μ-**stationary**, i.e., $\nu = \mu\nu$.

Let $S = S(\mu)$ be the semigroup generated by the support of the measure μ. Then for any point $g \in G$ the support of the measure type $[\delta_g\colon P_\mu]$ is gS. Thus, as it follows from Proposition 1.2.5, the harmonic measures ν_{g_1} and ν_{g_2}, $g_1, g_2 \in G$ are non-singular if and only if $g_1 S \cap g_2 S \neq \varnothing$, i.e., $g_1^{-1} g_2 \in SS^{-1}$. If $S = G$, then the measure μ is called **non-degenerate**. In terms of the corresponding random walk non-degeneracy of μ means that any two points in G communicate, so that in this case the measure ν is quasi-invariant and equivalent to the harmonic measure type $\boldsymbol{\nu}$ on the Poisson boundary Γ, and the operator P_μ satisfies condition (P′). See [KV] and [K7] for a detailed discussion of properties of the Poisson boundary of random walks on discrete groups.

5.2.2 Let μ^∞ be the product measure obtained by multiplying an infinite number of copies of the measure μ with the index set $\mathbb{Z}$. Then the measure ${}_{\widetilde{m}}\mathbf{P}^{\mathbb{Z}}$ in the path space $G^{\mathbb{Z}}$ is the image of the product $\widetilde{m} \times \mu^\infty$ under the map $(g, \{h_n\}) \mapsto \{x_n\}$ defined as

$$x_n = \begin{cases} g & , \quad n = 0 \\ g h_1 \dots h_n & , \quad n > 0 \\ g h_0^{-1} \dots h_{n+1}^{-1} & , \quad n < 0 \end{cases}$$

(so that $x_n = x_{n-1} h_n$). In other words, the initial position $g = x_0$ and the increments $h_n = x_{n-1}^{-1} x_n$ uniquely determine the path $\{x_n\}$.

5.2.3 THEOREM [K8], [BL]: *For any non-degenerate probability measure μ on a countable group G the action of the group G on the product of Poisson boundaries of the Markov operators P_μ and $P_{\check\mu}$ is G-ergodic.*

Proof: Let **bnd** and **bnd**ˇ be the maps from the bilateral path space to the Poisson boundaries Γ and $\check\Gamma$ of the operators P_μ and $P_{\check\mu}$, respectively. Since the measure μ is non-degenerate, operators P_μ and $P_{\check\mu}$ satisfy condition (P′), so that by Theorem 1.4.3 the image of the type of the measure $_{\tilde m}\mathbf{P}^{\mathbb{Z}}$ in the bilateral path space is equivalent to the product of harmonic measure types on the product of the Poisson boundaries Γ and $\check\Gamma$.

Let f be a G-invariant function on the product of the Poisson boundaries, and

$$F(x) = f\big(\mathbf{bnd}^{\check{}}(x), \mathbf{bnd}(x)\big)$$

be the corresponding function on the bilateral path space. Since the function f is G-invariant, the value $F(x)$ depends on the increments $h_n = x_{n-1}^{-1}x_n$ only. On the other hand, the function F is shift invariant, so that it determines a shift invariant function in the space of increments, which is impossible because of the ergodicity of the Bernoulli shift in the space of increments. ∎

5.2.4 THEOREM: *Let ξ be a G-invariant quotient of the Poisson boundary of the random walk (G,μ) on a countable group G, and φ be a non-negative measurable function on the quotient Γ^ξ such that the measure $\varphi\nu^\xi$ is μ-stationary. Then φ is constant a.e. with respect to the measure ν^ξ.*

Proof: For notational simplicity we shall consider here only the case when the quotient Γ^ξ is the boundary Γ itself (in the general situation the proof goes along the same lines).

Stationarity of the measure $\varphi\nu$ means that almost everywhere

$$\varphi(\gamma) = \frac{d(\varphi\nu)}{d\nu}(\gamma) = \frac{d\mu(\varphi\nu)}{d\nu}(\gamma) = \sum \frac{dg(\varphi\nu)}{d\nu}(\gamma)\,\mu(g)$$
$$= \sum \varphi(g^{-1}\gamma)\,\frac{dg\nu}{d\nu}(\gamma)\,\mu(g).$$

In other words, $f_\gamma(g) = \varphi(g^{-1}\gamma)$ is a harmonic function of the **conditional random walk** on G conditioned by the point γ, or, φ is a harmonic function of the Markov chain on the Poisson boundary Γ of the random walk (G,μ) with the transition probabilities

$$p(\gamma, g^{-1}\gamma) = \mu(g)\,\frac{dg\nu}{d\nu}(\gamma)\,.$$

This chain preserves the measure ν, because

$$\sum p(g\gamma,\gamma)\,d\nu(g\gamma) = \sum \mu(g)\,\frac{dg\nu}{d\nu}(g\gamma)\,d\nu(g\gamma) = d\nu(\gamma)\,,$$

and its reversed chain has the transition probabilities

$$\check{p}(\gamma, g\gamma) = p(g\gamma, g)\,\frac{d\nu(\gamma)}{d\nu(g\gamma)} = \mu(g)$$

(in other words, the reversed chain is the μ-**process** on Γ [Fu2]).

Thus, the unilateral path space of this chain can be obtained from the unilateral space of increments of the random walk (G,μ) by the formula

$$\gamma_n = \mathbf{bnd}(T^n h)\,.$$

In other words, γ_0 is the boundary point of the path $(e, h_1, h_1h_2, \dots)$, $\gamma_1 = h_1^{-1}\gamma_0$ is the boundary point of the path $(e, h_2, h_2h_3, \dots)$, and so on, where $h = (h_1, h_2, \dots)$ are sequences of i.i.d. increments h_n with distribution μ.

Consider the measurable set

$$A_t = \{\gamma \in \Gamma : \varphi(\gamma) \ge t\} \subset \Gamma \qquad t \ge 0\,,$$

and suppose that $\nu(A_t) > 0$. Then, as it follows from the Poincaré recurrence theorem applied to the shift in the space of increments, a.e. path $\{\gamma_n\}$ eventually hits the set A_t. Let τ be the corresponding stopping time, and $\tau \wedge n = \min\{\tau, n\}$. Then harmonicity of the function φ implies that

$$\varphi(\gamma) = \mathbf{E}_\gamma \varphi(\gamma_{\tau\wedge n}) \qquad \forall n \ge 0\,,$$

so that a.e.

$$\varphi(\gamma) \ge \mathbf{E}_\gamma \varphi(\gamma_\tau) \ge t\,,$$

and the function f is a.e. constant. ∎

5.2.5 THEOREM: *Let $\xi \preccurlyeq \zeta$ be two measurable G-invariant partitions of the Poisson boundary Γ of the random walk (G,μ) on a countable group G determined by a non-degenerate measure μ. Then the Radon–Nikodym cocycles of the measure types $\boldsymbol{\nu}^\xi$ and $\boldsymbol{\nu}^\zeta$ are cohomologous over the space Γ^ζ if and only if $\xi = \zeta$.*

Proof: Since the measure μ is non-degenerate, the measures ν^ξ and ν^ζ are equivalent to the measure types $\boldsymbol{\nu}^\xi$ and $\boldsymbol{\nu}^\zeta$, respectively, so that we can consider the Radon–Nikodym cocycles of the measures ν^ξ and ν^ζ. Their equivalence over the space Γ^ζ means that there is a measurable function φ on the space Γ^ζ such that a.e.

$$\frac{dg\nu^\xi}{d\nu^\xi}(\gamma^\xi) = \frac{dg\varphi\nu^\zeta}{d\varphi\nu^\zeta}(\gamma^\zeta) \qquad \forall g \in G .$$

Since the measure ν^ξ is μ-stationary, the measure $\varphi\nu^\zeta$ is also μ-stationary, so that $\varphi \equiv 1$ by Theorem 5.2.4, and a.e.

$$\frac{dg\nu^\xi}{d\nu^\xi}(\gamma^\xi) = \frac{dg\nu^\zeta}{d\nu^\zeta}(\gamma^\zeta) \qquad \forall g \in G .$$

Thus, the conditional walks determined by the quotients Γ^ξ and Γ^ζ coincide, so that $\xi = \zeta$. ∎

5.2.6 In the case when the measure μ has finite **entropy**

$$H(\mu) = -\sum_g \mu(g) \log \mu(g)$$

another proof of Theorem 5.2.5 can be obtained by using the notion of the **differential entropy**

$$E_\mu(\Gamma^\xi, \nu^\xi) = \int \log \frac{dh_1\nu}{d\nu}(\mathbf{bnd}^\xi(h))\, d\mu^\infty(h)$$

of the quotients (Γ^ξ, ν^ξ), where $\mathbf{bnd}^\xi$ is the map from the space of increments to the quotient Γ^ξ of the Poisson boundary by a G-invariant partition ξ [Fu2], [K1]. If $H(\mu) < \infty$, then

(a) The differential entropy $E_\mu(\Gamma^\xi, \nu^\xi)$ is finite for any partition ξ;

(b) $\frac{1}{n} \log \frac{dx_n\nu^\xi}{d\nu^\xi}(\mathbf{bnd}^\xi(h)) \to E_\mu(\Gamma^\xi, \nu^\xi)$ almost everywhere and in the space $L^1(\mu^\infty)$;

(c) $E_\mu(\Gamma^\xi, \nu^\xi) \le E_\mu(\Gamma^\zeta, \nu^\zeta)$ for any two partitions $\xi \preccurlyeq \zeta$, and the equality hold iff $\xi = \zeta$.

Thus, if $\xi \neq \zeta$, then almost everwhere

$$F_n(h) = \frac{1}{n}\left[\log \frac{dx_n\nu^\zeta}{d\nu^\zeta}(\mathbf{bnd}^\zeta(h)) - \log \frac{dx_n\nu^\xi}{d\nu^\xi}(\mathbf{bnd}^\xi(h))\right] \to \infty .$$

On the other hand, if the Radon–Nikodym cocycles of the measures ν^ξ and ν^ζ are equivalent, then there exists a measurable function f on the space of increments such that

$$F_n(h) = \frac{1}{n}\left(f(T^n h) - f(h)\right) ,$$

which is impossible by the Poincaré recurence theorem applied to the Bernoulli shift T in the space of increments.

5.2.7 THEOREM: *Let Γ be the Poisson boundary of the random walk (G,μ) determined by a probability measure μ on a countable group G. Then an element c from the center $\mathbb{Z}(G)$ of the group G acts on G trivially if and only if $c \in SS^{-1}$, where $S = S(\mu)$ is the subgroup generated by the support of the measure μ.*

Proof: As it follows from Proposition 1.2.5, the measures ν and $c\nu$ on the Poisson boundary are singular if and only if $c \notin SS^{-1}$. Thus, if $c \notin SS^{-1}$, then the action of c is non-trivial. Suppose that $c \in SS^{-1}$, so that the measures ν and $\check{\nu}$ are non-singular. Since $c \in Z(G)$,

$$\mu c\nu = c\mu\nu = c\nu,$$

so that the measure $c\nu$ is μ-stationary. Thus, the measure $\nu \wedge c\nu$ is also μ-stationary, which by Theorem 5.2.4 implies that $\nu \prec c\nu$. The same argument applied to c^{-1} implies that $\nu \prec c^{-1}\nu$, i.e., $c\nu \prec \nu$. Hence, $c\nu \sim \nu$, and by Theorem 5.2.4 $c\nu = \nu$. Thus, for ν-a.e. $\gamma \in \Gamma$ and any $g \in S$

$$\frac{dg\nu}{d\nu}(c^{-1}\gamma) = \frac{dgc\nu}{dc\nu}(\gamma) = \frac{dg\nu}{d\nu}(\gamma)\,,$$

so that the points $c^{-1}\gamma$ and γ determine the same harmonic functions on S, and $c^{-1}\gamma = \gamma$. ∎

Remark: The proof of this theorem is very close to the proof of a similar statement in [Fu3] (Theorem 11.2). See also [Gu1], [Gu2] for triviality of the Poisson boundary of random walks on nilpotent Lie groups. Note that the question about triviality of the Poisson boundary of the random walk (G,μ) on a general nilpotent Lie group G (without any absolute continuity assumptions on the measure μ) is still open.

References

[An] A. Ancona, *Théorie du potentiel sur les graphes et les variétés*, Springer Lecture Notes in Mathematics **1427** (1990), 4–112.

[Av] A. Avez, *Harmonic functions on groups*, in *Differential Geometry and Relativity* Reidel, Dordrecht, 1976, pp. 27–32.

[BL] W. Ballmann and F. Ledrappier, *The Poisson boundary for rank one manifolds and their cocompact lattices*, Forum Mathematicum **6** (1994), 301–313.

[BP] I. Benjamini and Y. Peres, *Random walks on a tree and capacity in the interval*, Annales de l'Institut Henri Poincaré. Probabilités et Statistiques **28** (1992), 557–593.

[Br] R. Brooks, *The fundamental group and the spectrum of the Laplacian*, Commentarii Mathematici Helvetici **56** (1981), 501–508.

[CFS] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai, *Ergodic Theory*, Springer, New York, 1982.

[De] Y. Derriennic, *Lois "zéro ou deux" pour les processus de Markov, applications aux marches aléatoires*, Annales de l'Institut Henri Poincaré. Probabilités et Statistiques **12** (1976), 111–129.

[Do] H. Donnelly, *Bounded harmonic functions and positive Ricci curvature*, Mathematische Zeitschrift **191** (1986), 559–565.

[Dy] E. B. Dynkin, *Markov Processes and Related Problems of Analysis*, London Mathematical Society Lecture Note Series 54, Cambridge University Press, Cambridge, 1982.

[Fo] S. R. Foguel, *Harris operators*, Israel Journal of Mathematics **33** (1979), 281–309.

[Fu1] H. Furstenberg, *A Poisson formula for semi-simple Lie groups*, Annals of Mathematics **77** (1963), 335–386.

[Fu2] H. Furstenberg, *Random walks and discrete subgroups of Lie groups*, in *Advances in Probability and Related Topics* Vol. 1, M. Dekker, New York, 1971, pp. 3–63.

[Fu3] H. Furstenberg, *Boundary theory and stochastic processes on homogeneous spaces*, in *Proceedings of Symposia in Pure Mathematics*, Vol. 26, AMS, Providence, R.I., 1973, pp. 193–229.

[Gr] F. P. Greenleaf, *Invariant Means on Topological Groups and their Applications*, Van Nostrand, New York, 1969.

[Gu1] Y. Guivarc'h, *Croissance polynomiale et périodes des fonctions harmoniques*, Bulletin de la Société Mathématique de France **101** (1973), 333–379.

[Gu2] Y. Guivarc'h, *Quelques propriétés asymptotique des produits de matrices aléatoires*, Springer Lecture Notes in Mathematics **774** (1980), 178–250.

[K1] V. A. Kaimanovich, *The differential entropy of the boundary of a random walk on a group*, Russian Mathematical Surveys **38:5** (1983), 142–143.

[K2] V. A. Kaimanovich, *Examples of non-commutative groups with non-trivial exit boundary,* Journal of Soviet Mathematics **28** (1985), 579–591.

[K3] V. A. Kaimanovich, *An entropy criterion for maximality of the boundary of random walks on discrete groups,* Soviet Mathematics. Doklady **31** (1985), 193–197.

[K4] V. A. Kaimanovich, *Brownian motion and harmonic functions on covering manifolds. An entropy approach,* Soviet Mathematics. Doklady **33** (1986), 812–816.

[K5] V. A. Kaimanovich, *Brownian motion on foliations: entropy, invariant measures, mixing,* Functional Analysis and its Applications **22** (1988), 326–328.

[K6] V. A. Kaimanovich, *Discretization of bounded harmonic functions on Riemannian manifolds and entropy,* in *Proceedings of the International Conference on Potential Theory, Nagoya, 1990* (M. Kishi, ed.), de Gruyter, Berlin, 1992, pp. 213–223.

[K7] V. A. Kaimanovich, *Measure-theoretic boundaries of Markov chains, 0-2 laws and entropy,* in *Proceedings of the International Conference on Harmonic Analysis and Discrete Potential Theory, Frascati, 1991* (M. A. Picardello, ed.), Plenum Press, New York, 1992, pp. 145–180.

[K8] V. A. Kaimanovich, *Bi-harmonic functions on groups,* Comptes Rendus de l'Academie des Sciences, Paris **314** (1992), 259–264.

[K9] V. A. Kaimanovich, *The Poisson boundary of hyperbolic groups,* Comptes Rendus de l'Academie des Sciences, Paris **318** (1994), 59–64.

[K10] V. A. Kaimanovich, *Erdogicity of the harmonic invariant measure of the geodesic flow on hyperbolic spaces,* Journal für die Reine und Angewandte Mathematik, (1994).

[Ka] F. I. Karpelevich, *The geometry of geodesics and the eigenfunctions of the Laplace–Beltrami operator on symmetric spaces,* Transactions of the Moscow Mathematical Society **14** (1965), 51–199.

[Ki] Y. Kifer, *Bounded harmonic functions on nonamenable covers of compact manifolds,* Israel Journal of Mathematics **61** (1988), 170–178.

[KL] Y. Kifer and F. Ledrappier, *Hausdorff dimension of harmonic measures on negatively curved manifolds,* Transactions of the American Mathematical Society **318** (1990), 685–704.

[KM] Yu. T. Kuz'menko and S. A. Molchanov, *Counterexamples to Liouville-type theorems,* Moscow University Mathematics Bulletin **34** (1979), 35–39.

[Kr] U. Krengel, *Ergodic Theorems,* de Gruyter, Berlin, 1985.

[KV] V. Kaimanovich and A. Vershik, *Random walks on discrete groups: boundary and entropy,* Annals of Probability **11** (1983), 457–490.

[KW] O. H. Kegel and B. A. F. Wehrfritz, *Locally Finite Groups,* North-Holland, Amsterdam, 1973.

[Li] V. Ya. Lin, *Liouville coverings of complex spaces, and amenable groups,* Mathematics of the USSR-Sbornik **60** (1988), 197–216.

[Lr] R. Lyons, *Random walks and percolation on trees,* Annals of Probability **18** (1990), 931–958.

[LS] T. Lyons and D. Sullivan, *Function theory, random paths and covering spaces,* Journal of Differential Geometry **19** (1984), 299–323.

[Lt] T. Lyons, *Instability of the Liouville property for quasi-isometric Riemannian manifolds and reversible Markov chains,* Journal of Differential Geometry **26** (1987), 33–66.

[LT] P. Li and L.-F. Tam, *Positive harmonic functions on complete manifolds with nonnegative curvature outside a compact set,* Annals of Mathematics **125** (1987), 171–207.

[Ma] G. A. Margulis, *Quotient groups of discrete subgroups and measure theory,* Functional Analysis and its Applications **12** (1978), 295–305.

[Mo] S. A. Molchanov, *On the Martin boundaries for the direct product of Markov chains,* Theory of Probability and its Applications **12** (1967), 307–310.

[N] P. J. Nicholls, *Ergodic Theory of Discrete Groups,* Cambridge Univ. Press, 1989.

[Paa] A. Patterson, *Amenability,* Math. Surveys & Monographs, Vol. 29, AMS, Providence, R.I., 1988.

[Pas] S. J. Patterson, *Lectures on measures on limit sets of Kleinian groups,* in *Analytical and Geometric Aspects of Hyperbolic Space* (D. B. A. Epstein, ed.), London Math. Soc. Lecture Note Series, Vol. 111, Cambridge Univ. Press, 1987, pp. 281–323.

[Pi] J.-P. Pier, *Amenable Locally Compact Groups,* Wiley, New York, 1984.

[PW1] M. A. Picardello and W. Woess, *Martin boundaries and random walks: ends of trees and groups,* Transactions of the American Mathematical Society **302** (1987), 185–205.

[PW2] M. A. Picardello and W. Woess, *Martin boundaries of Cartesian products of Markov chains,* Nagoya Mathematical Journal **128** (1992), 153–169.

[Ra] A. Raugi, *Fonctions harmoniques et théorèmes limites pour les marches aléatoires sur les groupes,* Mémoire de la Société Mathématique de France. Nouvelle Série **54** (1977), 5–118.

[Re] D. Revuz, *Markov Chains,* 2nd revised ed., North-Holland, Amsterdam, 1984.

[Ro] V. A. Rohlin, *On the fundamental ideas of measure theory,* Transactions of the American Mathematical Society **10** (1962), 1–54.

[Rob] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups,* Springer, Berlin, 1972.

[Sch] K. Schmidt, *Algebraic ideas in ergodic theory,* CBMS Regional Conference Series in Math., 76, Amer. Math. Soc., Providence, R.I., 1990.

[Su] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions,* Publications of Mathématiques de l'IHES **50** (1979), 171–202.

[Ta] M. Taniguchi, *Examples of discrete groups of motions conservative but not ergodic at infinity,* Ergodic Theory and Dynamical Systems **8** (1988), 633–636.

[To] D. Toledo, *Bounded harmonic functions on coverings,* Proceedings of the American Mathematical Society **104** (1988), 1218–1219.

[Tw] R. L. Tweedie, *R-Theory for Markov chains on a general state space I, II,* Annals of Probability **2** (1974), 840–878.

[V] J. Velling, *Recurrent Fuchsian groups whose Riemannian surfaces have infinite dimensional spaces of bounded harmonic functions,* Japan Academy. Proceedings. Series A. Mathematical Sciences **65** (1989), 211–214.

[VM] J. Velling and K. Matsuzaki, *The action at infinity of conservative groups of hyperbolic motions need not have atoms,* Ergodic Theory and Dynamical Systems **11** (1991), 577–582.

[Z1] R. J. Zimmer, *Amenable ergodic group actions and an application to Poisson boundaries of random walks,* Journal of Functional Analysis **27** (1978), 350–372.

[Z2] R. J. Zimmer, *Ergodic Theory and Semisimple Groups,* Birkhäuser, Boston, 1984.